Efficient estimation of power spectral density from laser Doppler anemometer data

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Abstract

A non-biased estimator of power spectral density (PSD) has been introduced for data obtained from a zeroth order interpolated LDA data set. The systematic error, sometimes referred to as the 'particle-rate' filter, is removed using an FIR filter parameterized using the mean particle rate. Independent from this, a procedure for estimating the measurement system noise is introduced and applied to the estimated spectra. The spectral estimation is performed in the domain of the autocorrelation function and assumes no further process parameters. The new technique is illustrated using measured LDA data with direct comparison to simultaneously acquired hot-wire data.

1 Introduction

Spectral estimation from laser Doppler anemometer (LDA) velocity data requires special consideration due to the randomly sampled nature of the signal in time. One commonly used approach is signal reconstruction, or interpolation, in which a data set with time equidistant samples is obtained through resampling of the interpolated signal. However the expectation of classical power spectral density (PSD) estimators may be strongly biased with systematic errors, which calls for a note of caution [6]. Among other effects, conventional interpolation techniques lead to an increased time correlation, equivalent to a low-pass filter. This "particle-rate" filter effect has been well studied for the case of sample-and-hold (S+H) reconstruction, i.e. zero order interpolation, and leads to PSD estimates which are valid only up to a frequency of $f < \dot{n}/2\pi$, where $\dot{n}$ is the mean particle rate [1]. For sparsely seeded flows this is a severe restriction and furthermore, even the valid spectral range may suffer from aliasing errors if power above this limit exists in the original data set. The fact that the filter characteristic introduced by the interpolation and resampling resembles closely typical turbulence spectra, only increases the danger of misinterpretation. More specifically, within the inertial subrange, the rate of dissipation can be inferred from the power spectral density [3]. A poorly estimated spectrum will therefore lead directly to improperly estimated dissipation rates.

In previous work the authors have shown that it is possible to estimate the systematic filter effect for one-point interpolation schemes, meaning schemes which use only the last valid data sample for interpolation within the current time interval. This would include the widely used S+H interpolation or the single exponential interpolation [5]. Once estimated, this bias can be removed from the spectral estimate, essentially corresponding to applying an FIR filter. This possibility exists due to the linear character of the particle-rate filter, which allows a matrix inversion. The remaining limit of power estimation resolution is dictated by the measurement system noise, which can be assumed to be white [4]. The noise contribution to the spectrum remains unchanged by applying an FIR filter. Some techniques, using a Kalman filter have been proposed to estimate and correct for this noise component, however these have always assumed a high particle rate and have not addressed the particle-rate filter or interpolation error [2, 8].

In the following work the error introduced by interpolation and the noise error are considered independently and removed from the PSD estimate. The technique for removing effects of the 'particle-rate' filter is briefly reviewed and some examples of its performance using LDA data taken from a free jet are presented. A new method of removing the noise component is then introduced and discussed in detail. Measurement data are used to illustrate the effectiveness of the method.

2 Correction for the Interpolation Error

The correction for the interpolation error introduced in [7] is applicable to LDA data sets obtained through one-point reconstruction and resampling, the most common being the sample and hold method. In the following work the sample and hold signal is resampled at regular time intervals, from which the autocorrelation function is computed.

$$R_{r;k} = R_r(\tau_k) = R_r(k/f_a) = \sum_{j=0}^{J-K} u_{r;j} u_{r;j+k} \qquad k = 0, \ldots, K-1 \tag{1}$$

where the subscript r stands for resampled and J is the largest integer number in the observation time. The power spectral density is given as

$$\begin{aligned} S_{r;j} &= S_r(f_j) = S_r\left(\frac{jf_a}{2K-1}\right) \\ &= \frac{1}{f_a} \sum_{k=0}^{2K-2} R_{r;k} e^{-2\pi ijk/(2K-1)} \qquad j = 0, \ldots, 2K-2 \end{aligned} \tag{2}$$

The result of this procedure is illustrated in figure 1 for three measured LDA data sets, with a comparison to hot wire (HW) measurements performed without seed particles. The data were taken from an axisymmetric free jet at a position of $x/D = 6$, $z/D = 3$ ($D = 5$ cm). The jet Reynolds number based on the jet outlet diameter and bulk velocity was 40 000, with an integral time scale at the measurement position of 2 ms. The mean velocity at this position was 8.5 m s^{-1} with turbulence level of 24 %. The dissipation rate, estimated as $\epsilon \approx q^3/l$, was 150 m^2 s^{-3} and the Kolmogorov length scale was estimated using $\eta = (\nu^3/\epsilon)^{1/4}$ to be 65 μm. The particle rate of the LDA data sets could be controlled through the particle seeding and was 7 100 Hz for figure 1 (a), 8 200 Hz for figure 1 (b) and 300 Hz for figure 1 (c). Furthermore the noise level could be increased substantially by misaligning the receiving optics and increasing the amplifier gain, as was done for the data processed in figure 1 (b). All data sets consisted of 100 000 samples.

In figure 1 (a) the spectral estimate shows good agreement with the hot-wire result (dashed line) up to a frequency of appoximately 1.5 kHz, where the hot-wire result indicates the end of the inertial subrange. The expected filter cut-off frequency in this case is $\dot{n}/2\pi \approx 1.1$ kHz, which explains the deviation of the two spectra at higher frequencies. The LDA result is dominated by the filter roll-off. In figure 1 (b) the spectral estimate is further contaminated by the added noise which is apparent already at frequencies above 200 Hz. Otherwise, the high frequency portion of the spectrum again exhibits a typical filter characteristic. Finally, in figure 1 (c), the consequence of a much lower particle rate becomes apparent. The filter cut-off frequency is $\dot{n}/2\pi \approx 50$ Hz, which means that almost the entire estimated spectrum has a systematic error and no longer resolves even a portion of the inertial subrange. For reference, a fitted line with slope $-5/3$ has been added to figure 1 (c) and corresponds closely to the hot-wire spectrum. Using the expression $\phi(f) = K\epsilon^{2/3}(2\pi f/u)^{-5/3}$ with $K = 0.5$ given by [3], the dissipation rate

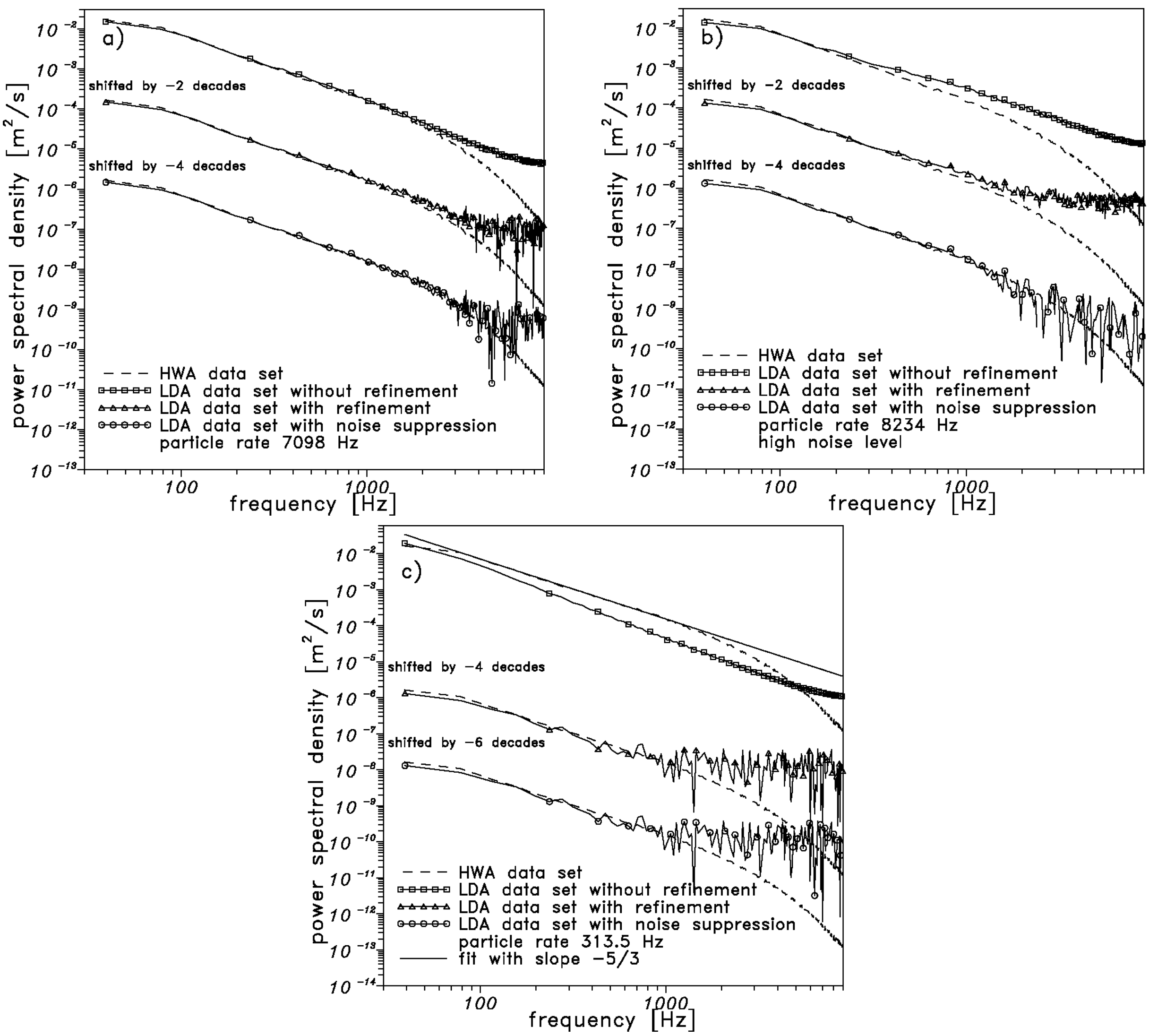

Figure 1: Power spectral density of LDA data sets without refinement, with refinement and with noise suppression

has been estimated to be 160 $\mathrm{m^2\,s^{-3}}$, which agrees remarkably well with the estimate given above.

The correction for the interpolation error begins with an expression for the autocorrelation function obtained from the interpolated and resampled data set. The derivation follows closely that of [1] and is given by [7]. For the case of the sample-and-hold interpolation, the following expression for the expectation is obtained.

$$\begin{aligned} E\{R_{r;k}\} &= R_{uu;0}\sum_{\zeta=-\infty}^{0}\left(1-e^{-\dot{n}}\right)e^{-\dot{n}(k-\zeta)}+\sum_{\xi=1}^{\infty}R_{uu;\xi}\sum_{\zeta=1}^{\min(k;\xi)}\left(1-e^{-\dot{n}}\right)^2 e^{-\dot{n}(k+\xi-2\zeta)} \\ &= e^{-\dot{n}k}\left\{R_{uu;0}+\frac{\left(e^{\dot{n}}-1\right)^2}{1-e^{2\dot{n}}}\sum_{\xi=1}^{\infty}e^{-\dot{n}\xi}\left(1-e^{2\dot{n}\min(k;\xi)}\right)R_{uu;\xi}\right\} \end{aligned} \quad (3)$$

The function $R_{uu;k}$ represents the true ACF of a continuous velocity signal, including the system noise. Equation (3) is a linear system which can be written as

$$E\{R_{r;k}\} = FR_{uu;k} \quad (4)$$

and inverted to yield a modified ACF, denoted by the subscript m

$$R_{m;k} = F^{-1} R_{r;k} \tag{5}$$

which is found to yield a non-biased estimate of $R_{uu;k}$ [7].

The function $R_{m;k}$ is a one-sided ACF with one sample less than $R_{r;k}$, because each value of $R_{m;k}$ depends on the subsequent value $R_{r;k+1}$. The values of $R_{m;k}$ are mirrored about the maximum coefficient to yield a symmetric function and the PSD is obtained through

$$\begin{aligned} S_{m;j} &= S_m(f_j) = S_m\left(\frac{jf_a}{2K-3}\right) \\ &= \frac{1}{f_a}\sum_{k=0}^{2K-4} R_{m;k} e^{-2\pi ijk/(2K-3)} \qquad j = 0, \ldots, 2K-4. \end{aligned} \tag{6}$$

Results of this procedure are also shown in figure 1 for the three experimental data sets. There are two prominent differences between the original spectra and those with refinement, both arising from the removal of the interpolation error. The first is, that the agreement between the LDA result and the HWA result extends to higher frequencies, especially apparent for the data taken at a low particle rate (figure 1 *c*), but also detectable for the case of a high particle rate (figure 1 *a*). The spectral estimate in figure 1 (*c*) appears to be reliable up to approximately 1 000 Hz, which is a twentyfold extension beyond the particle-rate filter cut-off frequency of 50 Hz applied in figure 1 (*c*). The second difference lies in an apparent increase of estimator variability at higher frequencies. This is deceptive, in fact the variability has not increased, but in the previous estimates of figure 1 the estimate at high frequencies was always dominated by the particle-rate filter. Using these estimates, which are now interpolation error free, one can also draw the conclusion that the estimator variance increases faster with decreasing particle rate than it decreases with increasing measurement duration.

3 Noise Suppression

Noise suppression has been attempted previously by [2] and [8] through the use of a Kalman filter. In this approach, a noise estimation was performed with respect to a prescribed model assuming that the noise component was spectrally white. However no account was made of the interpolation error. For this reason the technique is restricted to data sets with high particle rates, at which the interpolation error vanishes.

In the following approach the noise is also assumed to be uncorrelated, or white, i.e.

$$R_{m;k} = \begin{cases} R_{t;0} + \sigma_n^2 & \text{for } k = 0 \\ R_{t;k} & \text{for } k = 1, \ldots, K-2 \end{cases} \tag{7}$$

$$S_{m;j} = S_{t;j} + \sigma_n^2 / f_a \qquad j = 0, \ldots, 2K-4 \tag{8}$$

where f_a is the resample frequency, R_m and S_m are the ACF and the PSD respectively of the noise-containing signal and R_t and S_t are the same for the noise-free signal, i.e. the ACF and spectrum of the turbulent velocity fluctuations. Equation (8) indicates that the noise is additive to the spectrum, constant over all frequencies. If the noise is dominating, the spectrum is therefore constant.

The essence of noise suppression is to estimate σ_n^2 and subtract this from the individual spectral coefficients. A further assumption made in estimating the noise level is that the turbulent velocity fluctuations are *not* spectrally white over any frequency range. Thus, the distinguishing factor between signal and noise in the spectrum, is the flatness in frequency. A procedure is now proposed which operates on that portion of spectrum which is constant. After subtracting the

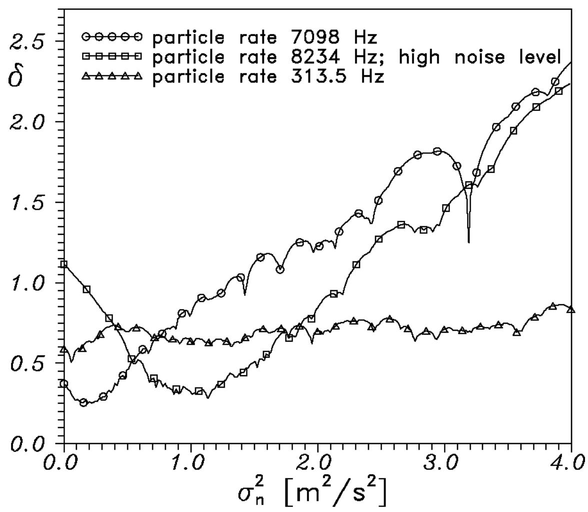

Figure 2: Scatter function δ as a function of postulated noise level σ_n^2

estimated noise level σ_n^2, the high frequency portion of the spectrum is not expected to remain constant, a feature which will be exploited to automate the estimation procedure.

Such an algorithm requires some specification of the portion of the spectrum to be considered, or alternatively, the entire spectrum can be used with a monotonic weighting which emphasizes the small amplitudes at high frequencies and considers with less weight the high amplitudes at low frequencies. Several weight functions fulfill these demands, however in the present work the natural logarithm of the *absolute* magnitude of the PSD has been chosen. This function increases sharply when the PSD amplitude approaches zero and furthermore can handle negative amplitudes, which invariably arise due to the variance of the spectral estimator and the resulting scatter above and below the true noise level. The following function is therefore considered.

$$\ln |S_{t;j}| = \ln \left| S_{m;j} - \frac{\sigma_n^2}{f_a} \right| \qquad j = 1, \ldots, 2K-4 \tag{9}$$

The optimal choice of σ_n^2 will lead to a maximum scatter or variance over all the values of the function given in Eq. (9). Therefore the estimation procedure consists of minimizing the scatter function

$$\delta = \left[\frac{1}{2K-5} \sum_{\substack{j=1 \\ j \neq \zeta}}^{2K-4} \left(\ln \left| S_{m;j} - \frac{\sigma_n^2}{f_a} \right| \right)^2 - \left(\frac{1}{2K-5} \sum_{\substack{j=1 \\ j \neq \zeta}}^{2K-4} \ln \left| S_{m;j} - \frac{\sigma_n^2}{f_a} \right| \right)^2 \right]^{-1} \tag{10}$$

with

$$\zeta \quad = \quad \arg \left\{ \left| S_{m;j} - \frac{\sigma_n^2}{f_a} \right| = \min \right\}$$

through the choice of σ_n^2. Note that the lowest value of the function given in Eq. (10), designated by the index ζ, is not used, to avoid any singularities in δ. Furthermore $S_{m;0}$ is not considered, since any bias error in the mean value could propagate disproportionally into the new spectral estimate.

Figure 2 illustrates the quantity δ as a function of σ_n^2 for the three data sets used in figure 1. In the case of high particle rates a distinct minimum of δ is observed. No such distinct minimum is seen in the case of low particle rate, indicating that the variance of the PSD estimate exceeds the noise level and no reliable estimate of σ_n^2 can be performed.

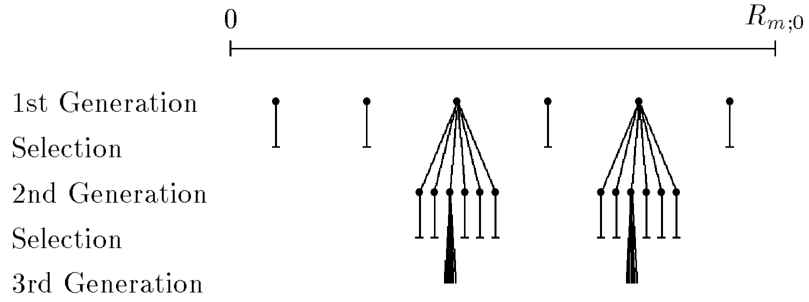

Figure 3: Search by generation branching

The noise estimation procedure reverts at this stage to a search by branching as described by the following. Initially δ is calculated at equal spaced values of σ_n^2 between zero and $R_{m;0}$ (maximum possible value). A new generation of smaller spaced σ_n^2 values is computed about the most promising values of the last generation, and so on, as indicated schematically in figure 3.

In the examples given in figure 2, 100 initial σ_n^2 values were used. At each of the 10 values corresponding to the lowest δ values, another 10 more densely spaced values of σ_n^2 were chosen, continuing for a total of 10 generations. At this stage the σ_n^2 value corresponding to a minimum of δ was subtracted from the PSD amplitudes at all frequencies.

The resulting PSD functions are superimposed in figure 1, shifted -4 decades from the original estimations. In the case of a high particle rate (figures 1 *a* and 1 *b*) the region of constant PSD has been completely eliminated. For the low noise data set the transition of the spectrum from the inertial subrange to the dissipative range has now been resolved. At the higher noise level (figure 1 *b*) no remnants of noise are left, leaving only the spectral fluctuations attributable to the estimator variance. The spectral estimate in figure 1 (*a*) and 1 (*b*) appear to be reliable up to frequencies of about 3 to 4 kHz, which is a fourfold increase beyond the particle-rate filter cut-off. At the low particle rate (figure 1 *c*) the estimator variance is dominant, as was previously remarked, so that the noise removal cannot be recognized.

4 Further Examples

Two further examples will be shown to underline the advantage of the new estimator. The first of these is a set of measurements taken from a free jet flow, in which very long data sets were recorded to illustrate the difference between noise and estimator variability. The mean data rate was approximately 4 000 Hz and two data sets consisting of 16 000 samples and 960 000 samples respectively were recorded. The corresoponding results of spectral estimation, with and without noise suppression are presented in the two diagrams of figure 4. In figure 4 (*b*) it is not immediately apparent whether the flattening of the spectrum taken from the short data set at higher frequencies is due to noise or whether this is due to a high variability of the estimate and appears flat due to the logarithmic presentation. The estimate in figure 4 (*b*), taken over 16 times the number of samples (long data set), clearly shows that the noise had been removed and that indeed, only the estimator variability was hindering a reliable spectral interpretation for the short data set.

The second example involves LDA data taken from a steady flow rig used to study valve inlet flows for internal combustion engines. For the present purpose two data sets with a particle rate of 300 Hz and 2 kHz respectively were used. The two data sets had the same number of velocity values. The data acquired at a high particle rate were processed using the conventional sample-and-hold reconstruction technique and the result was used as a reference spectrum which should be reliable up to approximately $2\,000/2\pi \approx 300$ Hz. The second data set with a low particle rate was then processed with and without the refinement and noise suppression step. The resample frequency was chosen to be 2 kHz.

The results are shown in figure 5. The reference spectrum exhibits a strong periodicity at 38 Hz associated with an unsteady swirl component and a second, weaker peak at about 90 Hz. The estimation at the particle rate of 300 Hz and without refinement (figure 5 *b*) misses this second peak completely, since the particle-rate filter has a cutoff frequency already at $300/2\pi \approx 50$ Hz. Also the "step noise" of this estimate [1] is seen clearly as an increased spectral level at low frequencies. On the other hand the new, refined estimate recovers the detail of the second peak even at the low particle rate of 300 Hz and agrees well with the reference spectrum up to a frequency of about 150 Hz. Above this value the variance of the estimator masks the true spectrum. Nonetheless, this represents a three fold increase in the frequency to which a reliable spectral estimate can be achieved.

5 Closing remarks

A spectral estimator for LDA data has been introduced which removes the interpolation error and the measurement system noise. The resulting spectrum is bias-free but displays a variance which increases with lower particle rates and increasing system noise. However this variance can be lowered to arbitrary levels by increasing the number of data samples. Thus this technique is also suitable for reliable estimation of power spectral density even at low particle rates. In this sense, the estimate is also alias-free and can in principle be extended to arbitrarily high frequencies, regardless of the mean particle rate.

Acknowledgments

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References

[1] Adrian, R. J. & Yao, C. S. 1987 Power Spectra of Fluid Velocities Measured by Laser Doppler Velocimetry. *Exp. in Fluids* **5**, 17–28.

[2] Benedict, L. H. & Gould, R. D. 1995 Experience Using Kalman Reconstruction for Enhanced Power Spectrum Estimates. In *Sixth Intl Conf. on Laser Anemometry* (ed. T. T. Huang; J. Turner; M. Kawahashi & M. V. Otugen). ASME, FED-Vol. 228, 1–7.

[3] Bradshaw, P. 1971 *An Introduction to Turbulence and its Measurement.* Pergamon Oxford.

[4] George, W. K. & Lumley, J. L. 1973 The laser Doppler velocimeter and its application to the measurement of turbulence. *J. Fluid Mech.* **60**, 321–362.

[5] Høst-Madsen, A. 1994 A New Method for Estimation of Turbulence Spectra for Laser Doppler Anemometry. In *Seventh Intl Symp. on Appl. of Laser Techn. to Fluid Mechanics* Lisbon, Portugal, paper 11.1.

[6] Müller, E.; Nobach, H. & Tropea, C. 1994 LDA Signal Reconstruction: Application to Moment and Spectral Estimation. In *Seventh Intl Symp. on Appl. of Laser Techn. to Fluid Mechanics* Lisbon, Portugal, paper 23.2.

[7] Nobach, H.; Müller, E. & Tropea, C. 1996 Refined Reconstruction Techniques for LDA Analysis. In *Eighth Intl Symp. on Appl. of Laser Techn. to Fluid Mechanics* Lisbon, Portugal, paper 36.2.

[8] Maanen, H. van & Tulleken, H. 1994 Application of Kalman Reconstruction to Laser-Doppler Anemometry Data for Estimation of Turbulent Velocity Fluctuations. In *Seventh Intl Symp. on Appl. of Laser Techn. to Fluid Mechanics* Lisbon, Portugal, paper 23.1.

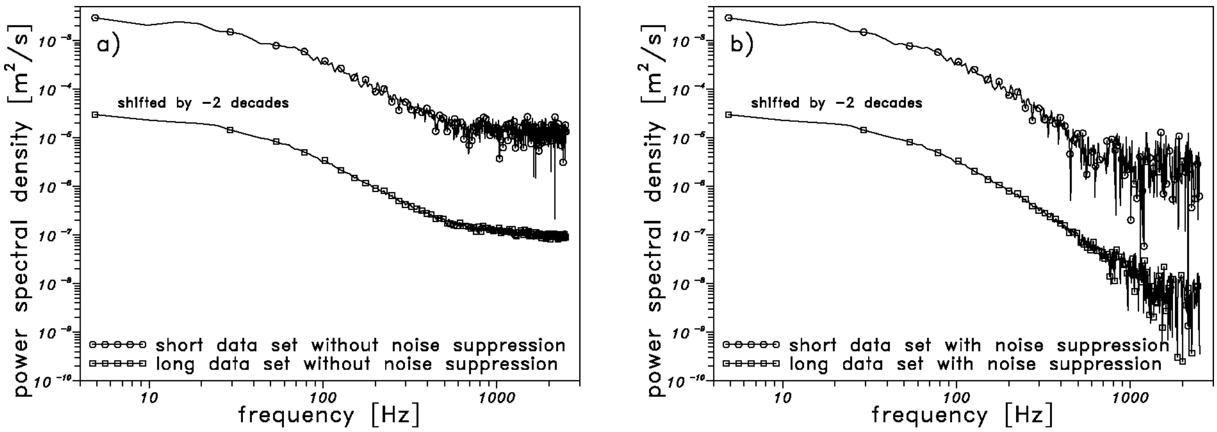

Figure 4: Example spectra (a) without and (b) with noise suppression

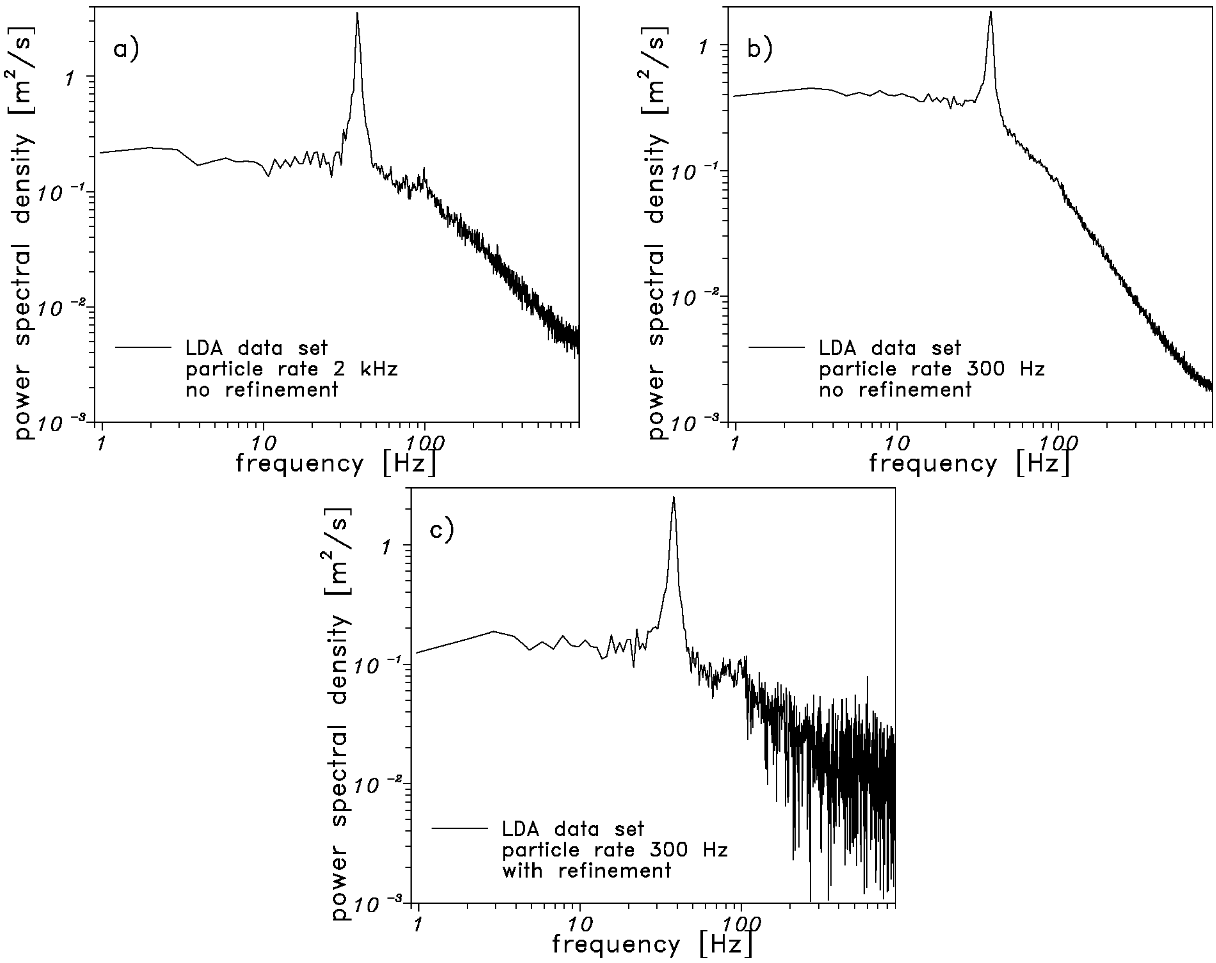

Figure 5: Example spectra taken from an unsteady swirl flow: (a) conventional S+H estimation at high particle rate; (b) conventional S+H estimate at low particle rate; (c) new estimation at low particle rate.