A direct estimator for cross-correlation and cross-spectral density estimation for two-channel laser Doppler anemometry

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April 25, 2016

Abstract

New estimators are introduced for correlation functions and cross-spectra between two channels of a laser Doppler velocimeter (LDV) based on direct spectral estimation. The estimators are applicable either to coincitent, two-component LDV data or to non-coincident, independent, two-point LDV data.

1 Introduction

For calculating the correlation function or the power spectral density from randomly sampled data from laser Doppler velocity measurements, estimation procedures are required, which consider the specific characteristics of LDV data, namely the sampling of the flow velocity at random arrival times, the data noise and the correlation of the sampling rate and the instantaneous velocity. Much effort has been put onto autocorrelation and auto-spectral estimators following thee different estimator classes, slot correlation, estimating a correlation function (correlogram) from the data [4, 7, 30, 31, 11, 12, 16, 18, 22, 24, 25, 26, 27, 29], direct spectral estimators, estimating a spectrum (periodogram) directly from the randomly sampled data [4, 5, 8, 9, 10, 19, 20, 32] and interpolation methods of the randomly sampled LDV data set yielding a continuous velocity over time, which then is re-sampled equidistantly with a given sampling frequency and processed by usual signal processing tools for equidistantly sampled data, including corrections of systematic errors [2, 13, 23, 28] and noise removal [21, 23].

Much less details are known about respective estimation procedures for two-channel data yielding the cross-correlation function and the cross-spectral density. So far detailed investigations exist about the following algorithms and applications.

- In [14] the possibility to use the slotting technique for the estimation of the cross-correlation function and the cross-spectrum is mentioned. There, no weighting mechanism has been realized, no local normalization, no fuzzy

slotting, and no investigation has been made about independent and dependent measurements between the channels. For autocorrelation, weighting schemes have been implemented [4], including the forward-backward inter-arrival-time weighting if transit times for individual weighting are not available [16, 18], local normalization and fuzzy slotting [30, 31, 22, 29] as well as Bessel's correction, if the data sets or data blocks are short and systematic errors due to the under-estimation of the block variance occur if the empirical block mean value is removed from the data blocks [19, 20].

- The application of the interpolation method to LDV cross-correlation and cross-spectra estimation has been investigated in [14] and [6]. Unfortunately, in both publications only special cases of two-channel measurements have been studied, namely either strictly independent or strictly coincident measurements in [6] or a mixture of only these two cases of measurements in [14]. The possibility of having a certain time delay of dependent measurements between the measurement channels has been mentioned in [14]. However, the there given procedures are valid only for a mixture of independent measurements and coincident dependent measurements between the channels, which is the case only if the respective measurement volumes of the two channels overlap. The method inherent weighting by holding the values longer if the data rate is lower can reduce the statistical bias due to the correlation between the instantaneous data rate and the velocity. At least at high data rates the suppression works efficient. Other, individual weighting schemes have not been realized for the interpolation method yet. Neither local normalization nor fuzzy slotting, originally developed for the slot correlation, have been adapted to the interpolation method so far. Bessel's correction, to suppress systematic errors due to the under-estimation of the velocity variances and velocity co-variance for short data sets or data blocks was not available at that time.

- The direct estimation has been used for the estimation of autocorrelation and auto-spectra only, including individual weighting [5, 9, 10, 32] or forward-backward inter-arrival-time weighting and Bessel's correction [19], local normalization and fuzzy time quantization [20]. Cross-correlation or cross-spectra have not been calculated with the direct estimation procedure so far.

- The direct estimation has also been used with quantized arrival times [5]. Quantized arrival times yield a quasi-equidistant data set with gaps with no data between the original samples. Filling these gaps with zeros yields an equidistant data set, which can be processed with common signal processing tools. This way either the correlation function can be calculated directly or the spectrum utilizing the fast Fourier transform. Both, the correlation function and the spectrum then are related through the Wiener-Khinchin theorem. This way the calculations for the direct estimation can be accelerated significantly. Since the time quantization
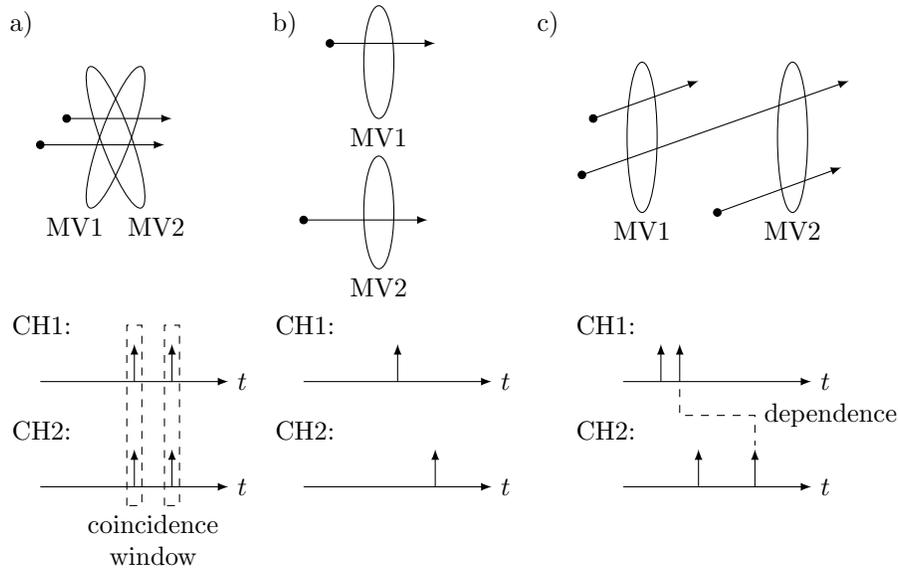
Figure 1: Fundamental sampling cases of two-channel laser Doppler data (CH1 and CH2) depending on the arrangement of the two measurement volumes MV1 and MV2: a) coincident measurements; b) independent measurements; c) mixed independent and dependent measurements with variation of the delay time between the two channels

> changes the results obtained, this method is counted as a fourth estimation type. It has not been used previously for the calculation of the cross-correlation or the cross-spectrum.

Unfortunately, the adaption of autocorrelation and auto-spectrum estimators to the two-channel case for randomly sampled LDV data is not as straight forward as for equidistantly sampled data. While for equidistant sampling, only one of two identical data sets in the autocorrelation/auto-spectrum calculation is replaced by a second data set, besides the always present irregular sampling and the correlation between the velocity and the data rate, additionally dependent and independent samples in the two channels must be considered [14] in the LDV case. Therefore, a detailed view into adequate estimation procedures is necessary for the three classes of estimation procedures given above.

2 Dependent and independent measurements

Two-channel data from multi-component or multi-point LDV systems can produce different sampling cases depending on the configuration of the system. These sampling characteristics may lead to different systematic errors. There-

fore, the following fundamental cases must be considered (Fig. 1).

- Coincident measurements: e.g. from two-component arrangements. The two channels are sampled together. This scheme yields a data set with identical sampling times $t_{1,i} = t_{2,i} = t_i$ and with identical number of samples $N_1 = N_2 = N$. This is most similar to the autocorrelation case. Estimation routines, errors and corrections are similar.

- Independent measurements: e.g. from transversal two-point measurements. The sampling of the two channels (number of samples and sampling times) is completely independent. This scheme yields a data set with no dependence between the two channels. Both, the number of samples and the sampling times of the two channels are independent of each other. The errors and corrections are different from the coincidence case.

- Mixed measurements: e.g. from two-component arrangements in free-running mode or from longitudinal two-point arrangements. There are both, independent measurements, from particles that are measured by only one of the two laser Doppler systems, and N_{dep} dependent measurements, from particles that cross both measurement volumes. In the latter case, the respective dependent samples $u_{\mathrm{d},1,i} = u(t_{\mathrm{d},1,i})$ and $u_{\mathrm{d},2,i} = u(t_{\mathrm{d},2,i})$ at arrival times $t_{\mathrm{d},1,i}$ and $t_{\mathrm{d},2,i}$ respectively with $i = 0 \ldots N_{\mathrm{dep}} - 1$ are subsets of the measured data $u_{1,i} = u(t_{1,i}), i = 0 \ldots N_1 - 1$ and $u_{2,j} = u(t_{2,j}), j = 0 \ldots N_2 - 1$. The samples of the two channels are time delayed $(t_{\mathrm{d},1,i} \neq t_{\mathrm{d},2,i})$, where the delay t_{d} may vary with the instantaneous velocity.

The direct spectral estimation has been realized only for the cases of coincident measurements or the non-coincident, but completely independent measurements. For the general case with both, depenedent and independent measurements, no appropriate estimator, based on the direct spectral estimation, has been developed yet.

3   The data sets

In the general case of two-channel data, two sets of irregularly sampled velocity data $u_{1,i} = u_1(t_{1,i})$ and $u_{2,j} = u_2(t_{2,j})$ at sampling times $t_{1,i}, i = 0 \ldots N_1 - 1$ and $t_{2,j}, j = 0 \ldots N_2 - 1$ are assumed together with individual weights $w_{1,i}$ and $w_{2,j}$ according to the velocity samples $u_{1,i}$ and $u_{2,j}$, e.g. the particle's transit times. If individual weights are not available, the inter-arrival times can be used for weighting, where both, the forward and the backward inter-arrival times may be necessary for the correlation and spectral estimations.

$$
\begin{aligned}
w_{\mathrm{bw},1,i} &= t_{1,i} - t_{1,i-1} \\
w_{\mathrm{fw},1,i} &= t_{1,i+1} - t_{1,i} \\
w_{\mathrm{bw},2,j} &= t_{2,j} - t_{2,j-1} \\
w_{\mathrm{fw},2,j} &= t_{2,j+1} - t_{2,j}
\end{aligned}
$$

To avoid that gaps in the data stream of experimental data lead to improperly large weights, as has been observed in experiments, all inter-arrival time weights derived from inter-arrival times larger than five times the mean inter-arrival time are set to zero. Due to this one looses only about 0.7% of useful data, while the outliers of large inter-arrival times are suppressed effectively.

In the general case of mixed independent and dependent data, in addition to the individual sampling by the independent measurements, there are pairs of dependent measurements $u_{1,\mathrm{dep},i}$ and $u_{2,\mathrm{dep},i}$ at sampling times $t_{1,\mathrm{dep},i}$ and $t_{2,\mathrm{dep},i}, i = 0 \ldots N_{\mathrm{dep}} - 1$ with the number N_{dep} of dependent measurements, where the sampling times can be coincident as for two-component arrangements with a coincidence window or different, with a varying delay as for the longitudinal two-point arrangement.

In the case of only coincident measurements, the two data sets have identical sampling at the times $t_{1,\mathrm{dep},i} = t_{2,\mathrm{dep},i}, i = 0 \ldots N_{\mathrm{dep}} - 1$. Since there are no indipendent measurements, it follows $N_1 = N_2 = N_{\mathrm{dep}}$. In this particular case, the individual weights $w_{1,\mathrm{dep},i}$ and $w_{2,\mathrm{dep},i}$ may be different or identical depending on the weighting scheme applied. In this case, also the forward- and backward inter-arrival times are identical for the two channels.

For completely independent sampling and inter-arrival time weighting, a differentiation of forward and backward inter-arrival times is not necessary. In this particular case, the weights can be chosen as

$$w_{\mathrm{bw},1,i} = w_{\mathrm{fw},1,i} = w_{1,i} \;\; = \;\; t_{1,i} - t_{1,i-1}$$
$$w_{\mathrm{bw},2,j} = w_{\mathrm{fw},2,j} = w_{2,i} \;\; = \;\; t_{2,j} - t_{2,j-1}$$

and all formulae become identical to those of the individual weighting.

Since for the direct spectral estimation method, no practical method of identification of dependent measurements from experimental data has been found so far, the data sets are assumed to be either completely independent or coincident. For the case of mixed dependent and independent samples, no practical algorithms has been realised, even if the following notation allows this for future developments.

4 Direct Spectral Estimator

The direct spectral estimation follows the derivations in [19], where all procedures have been adapted to the cross-correlation case.

4.1 Data pre-processing

The available data may be subdivided into blocks of a certain time duration T_{B} or the data may be obtained in blocks of a given record length. Due to the combination of Bessel's correction and the temporal limitation of the correlation function, both given below, the block duration can be chosen very flexible (compare [19]). It should be larger than the expected correlation interval of the flow and can be as large as the full data set.

From the data series $u_{1,i} = u_1(t_{1,i}), i = 0 \ldots N_1 - 1$ and $u_{2,i} = u_2(t_{2,i}), i = 0 \ldots N_2 - 1$ and the appropriate weigths $w_{1,i}$ and $w_{2,i}$, e.g. the transit times, one can calculate the block mean values as

$$\bar{u}_1 = \frac{\sum\limits_{i=0}^{N_1-1} w_{1,i} u_{1,i}}{\sum\limits_{i=0}^{N_1-1} w_{1,i}}$$

$$\bar{u}_2 = \frac{\sum\limits_{i=0}^{N_2-1} w_{2,i} u_{2,i}}{\sum\limits_{i=0}^{N_2-1} w_{2,i}}$$

or, using the backward inter-arrival times as given above

$$\bar{u}_1 = \frac{\sum\limits_{i=0}^{N_1-1} w_{\mathrm{bw},1,i} u_{1,i}}{\sum\limits_{i=0}^{N_1-1} w_{\mathrm{bw},1,i}}$$

$$\bar{u}_2 = \frac{\sum\limits_{i=0}^{N_2-1} w_{\mathrm{bw},2,i} u_{2,i}}{\sum\limits_{i=0}^{N_2-1} w_{\mathrm{bw},2,i}}$$

which is identical to the classical arrival-time weighting [3]. If the inter-arrival time between two samples exceeds a certain limit, the weighting factor can be set zero. Good experience has been obtained with a maximum value of five times the mean inter-arrival time. Due to this one looses about 0.7% of useful data, while the outliers of large inter-arrival times are suppressed effectively.

The block mean values then can be removed from the data to generate mean free data blocks for the following calculations of the cross-correlation function and the appropriate power spectral density. For the direct estimation, the computational costs increase with the square of the block length. Therefore, too large a block duration will be computational costly.

4.2   Estimation of the initial functions

The derivation of the initial spectra and correlation functions follows [19], adpated to the cross-correlation case.

First, direct spectral estimates of the weighted velocities and of the weights themselfes are derived. For individual weighting (e.g. transit-time weighting)

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these are

$$
\begin{aligned}
S'_{u,12}(f) &= \frac{T_{\mathrm{B}}}{W} \left[\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} w_{1,i} w_{2,j} u_{1,i} u_{2,j} \mathrm{e}^{-2\pi \mathtt{i} f (t_{2,j}-t_{1,i})} \right. \\
&\qquad \left. - \sum_{i=0}^{N_{\mathrm{dep}}-1} w_{1,\mathrm{dep},i} w_{2,\mathrm{dep},i} u_{1,\mathrm{dep},i} u_{2,\mathrm{dep},i} \mathrm{e}^{-2\pi \mathtt{i} f (t_{2,\mathrm{dep},i}-t_{1,\mathrm{dep},i})} \right] \\
&= \frac{T_{\mathrm{B}}}{W} \left[U_1^*(f) U_2(f) - U_{12,\mathrm{dep}}(f) \right] \\
S'_{w,12}(f) &= \frac{T_{\mathrm{B}}}{W} \left[\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} w_{1,i} w_{2,j} \mathrm{e}^{-2\pi \mathtt{i} f (t_{2,j}-t_{1,i})} \right. \\
&\qquad \left. - \sum_{i=0}^{N_{\mathrm{dep}}-1} w_{1,\mathrm{dep},i} w_{2,\mathrm{dep},i} \mathrm{e}^{-2\pi \mathtt{i} f (t_{2,\mathrm{dep},i}-t_{1,\mathrm{dep},i})} \right] \\
&= \frac{T_{\mathrm{B}}}{W} \left[W_1^*(f) W_2(f) - W_{12,\mathrm{dep}}(f) \right]
\end{aligned}
$$

with

$$
U_1(f) = \mathrm{DFT}\{w_{1,i} u_{1,i}\} = \sum_{i=0}^{N_1-1} w_{1,i} u_{1,i} \mathrm{e}^{-2\pi \mathtt{i} f t_{1,i}}
$$

$$
U_2(f) = \mathrm{DFT}\{w_{2,i} u_{2,i}\} = \sum_{i=0}^{N_2-1} w_{2,i} u_{2,i} \mathrm{e}^{-2\pi \mathtt{i} f t_{2,i}}
$$

$$
U_{12,\mathrm{dep}}(f) = \sum_{i=0}^{N_{\mathrm{dep}}-1} w_{1,\mathrm{dep},i} w_{2,\mathrm{dep},i} u_{1,\mathrm{dep},i} u_{2,\mathrm{dep},i} \mathrm{e}^{-2\pi \mathtt{i} f (t_{2,\mathrm{dep},i}-t_{1,\mathrm{dep},i})}
$$

$$
W_1(f) = \mathrm{DFT}\{w_{1,i}\} = \sum_{i=0}^{N_1-1} w_{1,i} \mathrm{e}^{-2\pi \mathtt{i} f t_{1,i}}
$$

$$
W_2(f) = \mathrm{DFT}\{w_{2,i}\} = \sum_{i=0}^{N_2-1} w_{2,i} \mathrm{e}^{-2\pi \mathtt{i} f t_{2,i}}
$$

$$
W_{12,\mathrm{dep}}(f) = \sum_{i=0}^{N_{\mathrm{dep}}-1} w_{1,\mathrm{dep},i} w_{2,\mathrm{dep},i} \mathrm{e}^{-2\pi \mathtt{i} f (t_{2,\mathrm{dep},i}-t_{1,\mathrm{dep},i})}
$$

and

$$
W = W_1^*(0) W_2(0) - W_{12,\mathrm{dep}}(0).
$$

Note that all N_{dep} dependent measurements are included in both, the N_1 measurements in channel 1 and the N_2 measurements in channel 2.

The above notation is valid for all three cases of sampling, the coincident sampling, the independent as well as the mixed sampling with independent

and dependent measurements. For coincident measurements all samples are dependent and $N_1 = N_2 = N_{\text{dep}}$, $w_{1,\text{dep},i} \equiv w_{1,i}$, $w_{2,\text{dep},i} \equiv w_{2,i}$. In this case the correction can be realized by using all measured samples in the appropriate correction sums $U_{12,\text{dep}}(f)$ and $W_{12,\text{dep}}(f)$. For independent measurements is $N_{\text{dep}} = 0$, and therefore no correction is required, the appropriate sums are zero. Only the mixed sampling with independent and dependent measurements is difficult to realize practically, because no unique identification of the dependent measurements from measured data has been found so far.

For forward-backward inter-arrival time weighting the primary spectral estimates are

$$
\begin{aligned}
S'_{u,12}(f) &= \frac{T_\text{B}}{W} \left[U_{12}(f) + U'_{12}(f) \right] \\
S'_{w,12}(f) &= \frac{T_\text{B}}{W} \left[W_{12}(f) + W'_{12}(f) \right]
\end{aligned}
$$

with

$$
\begin{aligned}
U_{12}(f) &= \sum_{\substack{i=0 \\ }}^{N_1-1} \sum_{\substack{j=0 \\ t_{2,j} < t_{1,i} + t_\text{d}(t_{1,i})}}^{N_2-1} w_{\text{fw},1,i} w_{\text{bw},2,j} u_{1,i} u_{2,j} \mathrm{e}^{-2\pi \mathrm{i} f(t_{2,j} - t_{1,i})} \\
&= \sum_{i=0}^{N_1-1} w_{\text{fw},1,i} u_{1,i} \mathrm{e}^{+2\pi \mathrm{i} f t_{1,i}} \left(\sum_{\substack{j=0 \\ t_{2,j} < t_{1,i} + t_\text{d}(t_{1,i})}}^{N_2-1} w_{\text{bw},2,j} u_{2,j} \mathrm{e}^{-2\pi \mathrm{i} f t_{2,j}} \right) \\
U'_{12}(f) &= \sum_{\substack{i=0 \\ }}^{N_1-1} \sum_{\substack{j=0 \\ t_{2,j} > t_{1,i} + t_\text{d}(t_{1,i})}}^{N_2-1} w_{\text{bw},1,i} w_{\text{fw},2,j} u_{1,i} u_{2,j} \mathrm{e}^{-2\pi \mathrm{i} f(t_{2,j} - t_{1,i})} \\
&= \sum_{j=0}^{N_2-1} w_{\text{fw},2,j} u_{2,j} \mathrm{e}^{-2\pi \mathrm{i} f t_{2,j}} \left(\sum_{\substack{i=0 \\ t_{1,i} < t_{2,j} - t_\text{d}(t_{2,j})}}^{N_1-1} w_{\text{bw},1,i} u_{1,i} \mathrm{e}^{+2\pi \mathrm{i} f t_{1,i}} \right)
\end{aligned}
$$

$$W_{12}(f) = \sum_{\substack{i=0}}^{N_1-1} \sum_{\substack{j=0 \\ t_{2,j}<t_{1,i}+t_\mathrm{d}(t_{1,i})}}^{N_2-1} w_{\mathrm{fw},1,i} w_{\mathrm{bw},2,j} \mathrm{e}^{-2\pi\mathrm{i}f(t_{2,j}-t_{1,i})}$$

$$= \sum_{i=0}^{N_1-1} w_{\mathrm{fw},1,i} \mathrm{e}^{+2\pi\mathrm{i}ft_{1,i}} \left(\sum_{\substack{j=0 \\ t_{2,j}<t_{1,i}+t_\mathrm{d}(t_{1,i})}}^{N_2-1} w_{\mathrm{bw},2,j} \mathrm{e}^{-2\pi\mathrm{i}ft_{2,j}} \right)$$

$$W'_{12}(f) = \sum_{\substack{i=0}}^{N_1-1} \sum_{\substack{j=0 \\ t_{2,j}>t_{1,i}+t_\mathrm{d}(t_{1,i})}}^{N_2-1} w_{\mathrm{bw},1,i} w_{\mathrm{fw},2,j} \mathrm{e}^{-2\pi\mathrm{i}f(t_{2,j}-t_{1,i})}$$

$$= \sum_{j=0}^{N_2-1} w_{\mathrm{fw},2,j} \mathrm{e}^{-2\pi\mathrm{i}ft_{2,j}} \left(\sum_{\substack{i=0 \\ t_{1,i}<t_{2,j}-t_\mathrm{d}(t_{2,j})}}^{N_1-1} w_{\mathrm{bw},1,i} \mathrm{e}^{+2\pi\mathrm{i}ft_{1,i}} \right)$$

and

$$W = W_{12}(0) + W'_{12}(0).$$

Note, that for each of the four expressions for $U_{12}(f)$, $U'_{12}(f)$, $W_{12}(f)$, $W'_{12}(f)$, the second sum is one addend of the first, outer sum. Note further, that the sums are not independent of each other, which means that the Fourier transform is not complete and direct calculations of the sums are necessary. Luckily, the sums can be calculated parallel, such that only the individual samples must be processed instead of all pairs of samples. If the forward or backward inter-arrival time between two samples exceeds a certain limit, the weigthing factor should be set zero. Good experience has been obtained with a maximum value of five times the mean inter-arrival time. Due to this one looses about $0.7\,\%$ of useful data, while the outliers of large inter-arrival times are suppressed effectively. The equations for the forward-backward inter-arrival time weighting can be re-used for individual weighting schemes (e.g. transit-time weighting), if the forward as well as the backward weights are replaced by the individual weights $(w_{\mathrm{bw},i} = w_{\mathrm{fw},i} = w_i)$.

The notation for the forward-backward inter-arrival time weighting is also valid for all three cases of sampling, the coincident sampling, the independent as well as the mixed sampling with independent and dependent measurements. For coincident measurements all samples are dependent with $t_\mathrm{d} \equiv 0$. In this case the relation of sampling times $t_{1,i} < t_{2,j}$, $t_{1,i} = t_{2,j}$ or $t_{1,i} > t_{2,j}$ can be identified by the relation of the indices $i < j$, $i = j$ or $i > j$. For independent measurements, the forward and backward weights can be chosen identical, e.g. the backward inter-arrival weights, as the classical arrival-time weighting scheme. In this particular case, the above formulae become first independent of a specific delay time t_d and second identical to the formulae for the individual weighting, e.g. the transit time weighting scheme, with the weights beeing the backward inter-arrival times. Only the mixed sampling with independent and

dependent measurements is difficult to realize practically, because no unique identification of the dependent measurements or the varying delay time $t_{\mathrm{d}}(t_{1,i})$ or $t_{\mathrm{d}}(t_{2,j})$ from measured data has been found so far.

Summarizing, for both, the transit-time weighting or other individual weighting schemes as well as for the forward-backward inter-arrival weighting, the identification of dependent measurements between the two channels is possible for coincident measurements and for fully independent measurements. For mixed measurements with independent and dependent measurements, no unique solution has been found to identify the dependent measurements from the data streams. So the subtraction of the contribution of the dependent measuremnts to the above sums cannot be realized, at least for measured data where no reliable information about the dependence is available. Unfortunately, in contrast to the slot correlation method, with the direct spectral estimation method, the following normalization in the correlation space also fails if a mixture of dependent and independent measurements occurs with a preferred delay time different than zero. Therefore, for the two special cases, the coincident measurements and the fully independent measurements, different direct spectral estimators have been realized, adapted to the specific cases. For the mixed independent and dependent data case, unfortunately, no universal estimation algorithm could be found.

The following processing steps, normalization and Bessel's correction, require the primary spectral estimates to be transformed into correlation functions. While the primary spectral estimates above can be calculated for any frequency f, the transformation into correlation functions requires the definition of a fundamental frequency F, which defines the temporal resolution $\Delta\tau = {}^{1}/{F}$ of the obtained correlation functions. The spectra as given above then must be calculated for the frequencies $f_j = j\Delta f, j = -\lceil T_{\mathrm{B}}F\rceil \dots \lceil T_{\mathrm{B}}F\rceil - 1$ with $\Delta f = {}^{1}/{2T_{\mathrm{B}}}$ and the cross-correlation functions can be obtained using the inverse DFT (IDFT)

$$R'_{u,12}(\tau_k) \;\; = \;\; F\,\mathrm{IDFT}\left\{S'_{u,12}(f_j)\right\} = \frac{1}{2T_{\mathrm{B}}} \sum_{j=-\lceil T_{\mathrm{B}}F\rceil}^{\lceil T_{\mathrm{B}}F\rceil - 1} S'_{u,12}(f_j)\mathrm{e}^{2\pi\mathrm{i}\tau_k f_j}$$

$$R'_{w,12}(\tau_k) \;\; = \;\; F\,\mathrm{IDFT}\left\{S'_{w,12}(f_j)\right\} = \frac{1}{2T_{\mathrm{B}}} \sum_{j=-\lceil T_{\mathrm{B}}F\rceil}^{\lceil T_{\mathrm{B}}F\rceil - 1} S'_{w,12}(f_j)\mathrm{e}^{2\pi\mathrm{i}\tau_k f_j}$$

At the same time the length of the correlation function can be reduced to a total length of $T_{\mathrm{C}} = K\Delta\tau$, where $\tau_k = k\Delta\tau, k = -\lfloor {}^{K}/{2}\rfloor \dots \lfloor {}^{(K-1)}/{2}\rfloor$, as a means to reduce the estimation variance of the final spectral density [17, 19] derived from the correlation function after the following normalization and Bessel's correction.

4.3 Normalization, Bessel's correction and final transformation

The final estimate of the correlation function is obtained by normalization as

$$R_{12}(\tau_k) = \frac{R'_{u,12}(\tau_k)}{R'_{w,12}(\tau_k)} + c_B$$

including Bessel's correction, where the correction c_B is related to the estimated variances of the mean estimators above. In the case of the direct estimation c_B is obtained similar to the procedure given in [19] adpated to the cross-correlation case as

$$c_B = \frac{\sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R'_{u,12}(\tau_k)}{T_{\mathrm{B}}F - \sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R'_{w,12}(\tau_k)}.$$

The final cross-correlation estimate can then be transformed by means of the discrete Fourier transform (DFT) to a power spectral density

$$S_{12}(f_j) = \Delta\tau \mathrm{DFT}\left\{R_{12}(\tau_k)\right\} = \Delta\tau \sum_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R_{12}(\tau_k)\mathrm{e}^{-2\pi\mathrm{i}f_j\tau_k}$$

with $f_j = j\Delta f, j = -\lfloor K/2 \rfloor \ldots \lfloor (K-1)/2 \rfloor$ giving a frequency resolution of $\Delta f = 1/K\Delta\tau$.

4.4 Remarks

Estimators of the cross correlation function and the cross spectral density for irregularly sampled two-channel data have been derived for two special cases, for coincident measurements and for fully independent measurements. For mixed independent and dependent measurements, especially for the case that the preffered delay time is different than zero, the developed procedure depends on a unique identification of the dependent measurements. Since no practical realization of a unique identification of the dependent measurements has been found, an appropriate correction of the respective sums is not possible. Note that this fact is similar to the slotting technique for estimating the cross correlation function and the cross spectral density. Unfortunately, in contrast to the slotting technique, where the influence of the different probabilities of dependent and independent measurements has been corrected using different weighting exponents depending on the fraction of dependent measurements in a given slot, no appropriate solution has been found for the direct estimation. In addition to this error, the estimates of the primary spectra show long-lasting oscillations if the preffered delay time is different than zero. If these spectral oscillation don't fit into the spectral range, also the correlation function shows oscillations as a truncations error of the appropriate spectrum, even in combination with

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the normalization. This error can be avoided by quantizing the arrival times of all measurements using the temporal resolution of the aspired correlation function. However, the error due to the different probabilities of dependent and independent measurements remains. A statistical bias due to the correlation of the velocity and the instantaneous data rate are suppressed due to the implementation of the weighting schemes. Example programs for the the two cases of either coincident data or fully independent data can be found at [1] including extensions of the local normalization and the fuzzy slotting [20] adapted to the cross-correlation case.

References

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