# A slotting technique for autocorrelation and spectral density estimation from laser Doppler anemometry data including fuzzy slotting and local normalization

Holger Nobach

December 3, 2016

**Abstract**

The slotting technique for calculating the autocorrelation function and the spectrum from laser Doppler data is revisited and extended by recently developed processing steps.

## 1 Introduction

For calculating the correlation function or the power spectral density from randomly sampled data from laser Doppler velocity measurements, estimation procedures, which consider the specific characteristics of LDV data are required, namely the sampling of the flow velocity at random arrival times, the data noise and the correlation of the sampling rate and the instantaneous velocity. Much effort has been put onto autocorrelation and autospectral estimators following three different estimator classes, slot correlation, estimating a correlation function (correlogram) from the data [3, 26, 27, 9, 10, 13, 14, 18, 20, 21, 22, 23, 25], direct spectral estimators, estimating a spectrum (periodogram) directly from the randomly sampled data [3, 4, 5, 6, 7, 8, 15, 16, 28] and interpolation methods of the randomly sampled LDV data set yielding a continuous velocity over time, which then is re-sampled equidistantly with a given sampling frequency and processed by usual signal processing tools for equidistantly sampled data, including corrections of systematic errors [2, 11, 19, 24] and noise removal [17, 19]. A fourth processing method, using quantitized arrival times, has not achieved much attention so far. It has been used before, in [12], however, it is broadly available only after publication of [4], where it has been used to accelerate the direct estimation of auto-spectra in combination with a normalization of the appropriate correlation function.

## 2 The data set

In the following processing steps a sets of irregularly sampled velocity data $u_i = u(t_i)$ at sampling times $t_i, i = 0 \ldots N-1$ is assumed together with individual weights $w_i$ according to the velocity smaples $u_i$, e.g. the particle's transit times. If individual weights are not available, the inter-arrival times can be used for weighting, where both, the forward and the backward inter-arrival times are necessary for the correlation and spectral estimations.

$$\begin{aligned} w_{\mathrm{bw},i} &= t_i - t_{i-1} \\ w_{\mathrm{fw},i} &= t_{i+1} - t_i \end{aligned}$$

To avoid that gaps in the data stream of experimental data lead to improperly large weights, as has been observed in experiments, all inter-arrival time weights derived from inter-arrival times larger than five times the mean inter-arrival time are set to zero. Due to this one looses only about $0.7\,\%$ of useful data, while the outliers of large inter-arrival times are suppressed effectively.

## 3 Slotting Technique

The slotting technique is reviewed including individual data weighting (e.g. transit-time weighting) or forward-backward inter-arrival-time weighting [13, 14] as an alternative, if reliable estimates of the particle transit times are not available. To suppress the influence of data noise and to avoid varying probability densities of pairs of data, no self-products are used (see [20]). Local normalization [27, 25] and fuzzy slotting [18] are introduced as optional extensions of the base algorithm. The Python code of the estimator is available at [1].

### 3.1 Base Algorithm

The slotting technique derives the autocorrelation function $R(\tau_k)$ at discrete time lags $\tau_k = k\Delta\tau$ by averaging the products of all data pairs $u_i$ and $u_j$ falling into bins of the width $\Delta\tau$. For the base slotting technique with individual data weighting or forward-backward inter-arrival-time (FBAT) weighting, without self-products and with Bessel's correction this is

$$R(\tau_k) = \frac{R_u(\tau_k)}{R_w(\tau_k)} + \sigma_{\bar{u}}^2, \tag{1}$$

where $\sigma_{\bar{u}}^2$ is the estimate of the variance of the mean estimator (an appropriate estimator follows) and

$$R_u(\tau_k) = \sum_{\substack{i=0 \\ i\neq j}}^{N-1} \sum_{j=0}^{N-1} \left\{ \begin{array}{ll} w_{\mathrm{bw},i} w_{\mathrm{fw},j} & \text{if } t_i < t_j \\ w_{\mathrm{fw},i} w_{\mathrm{bw},j} & \text{if } t_i > t_j \end{array} \right\} u_i u_j b_k(t_j - t_i) \tag{2}$$

$$R_w(\tau_k) = \sum_{\substack{i=0 \\ i\neq j}}^{N-1} \sum_{j=0}^{N-1} \left\{ \begin{array}{ll} w_{\mathrm{bw},i} w_{\mathrm{fw},j} & \text{if } t_i < t_j \\ w_{\mathrm{fw},i} w_{\mathrm{bw},j} & \text{if } t_i > t_j \end{array} \right\} b_k(t_j - t_i) \tag{3}$$

with

$$b_k(\Delta t) = \left\{ \begin{array}{ll} 1 & \text{for } |\Delta t - k\Delta\tau| < {}^{\Delta\tau}\!/_{2} \\ 0 & \text{otherwise} \end{array} \right. . \tag{4}$$

Note that for the slotting technique $R_u(\tau_k)$ and $R_w(\tau_k)$ are the raw sums of cross-products. For forward-backward inter-arrival time weighting (FBAT), the weights read as

$$w_{\mathrm{bw},i} = t_i - t_{i-1} \tag{5}$$

and

$$w_{\mathrm{fw},i} = t_{i+1} - t_i. \tag{6}$$

Here also, a maximum weight of five times the mean inter-arrival time is accepted to avoid errors due to gaps in the data stream of experimental data. All values beyond are set to zero. If other weighting schemes with individual data weights $w_i$ are used, then $w_{\mathrm{bw},i}$ and $w_{\mathrm{fw},i}$ can be replaced by these individual weights $w_i$, e.g. the transit times for the well correcting transit time weighting (TT).

The maximum time lag is typically chosen smaller than the duration of the measurement. With a chosen temporal resolution of the correlation function $\Delta\tau$ and a number of samples $K$, the correlation function will be estimated for $k = -\lfloor {}^{K}\!/_{2} \rfloor \ldots \lfloor {}^{(K-1)}\!/_{2} \rfloor$ and then be transformed by means of the discrete Fourier transform (DFT) to a power spectral density

$$S(f_j) = \Delta\tau \cdot \mathrm{DFT}\left\{R(\tau_k)\right\} = \Delta\tau \sum_{k=-\lfloor {}^{K}\!/_{2} \rfloor}^{\lfloor {}^{(K-1)}\!/_{2} \rfloor} R(\tau_k) \mathrm{e}^{-2\pi \mathrm{i} f_j \tau_k} \tag{7}$$

with $f_j = j\Delta f, j = -\lfloor {}^{K}\!/_{2} \rfloor \ldots \lfloor {}^{(K-1)}\!/_{2} \rfloor$ giving a frequency resolution of $\Delta f = {}^{1}\!/_{K\Delta\tau}$.

## 3.2 Local Normalization

With local normalization [26, 27, 25], an estimate of the correlation coefficient function is derived as

$$\rho(\tau_k) = \frac{R_u(\tau_k) + \sigma_{\bar{u}}^2 R_w(\tau_k)}{\sqrt{\left[R_{u_1}(\tau_k) + \sigma_{\bar{u}}^2 R_w(\tau_k)\right] \left[R_{u_2}(\tau_k) + \sigma_{\bar{u}}^2 R_w(\tau_k)\right]}}, \tag{8}$$

where $\sigma_{\bar{u}}^2$ is the estimate of the estimation variance of the mean estimator (an appropriate estimator follows) and $R_u(\tau_k)$ and $R_w(\tau_k)$ as before and additionally

$$R_{u_1}(\tau_k) = \sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ i \neq j}}^{N-1} \left\{ \begin{array}{ll} w_{\mathrm{bw},i} w_{\mathrm{fw},j} & \text{if } t_i < t_j \\ w_{\mathrm{fw},i} w_{\mathrm{bw},j} & \text{if } t_i > t_j \end{array} \right\} u_i^2 b_k(t_j - t_i) \tag{9}$$

$$R_{u_2}(\tau_k) = \sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ i \neq j}}^{N-1} \left\{ \begin{array}{ll} w_{\mathrm{bw},i} w_{\mathrm{fw},j} & \text{if } t_i < t_j \\ w_{\mathrm{fw},i} w_{\mathrm{bw},j} & \text{if } t_i > t_j \end{array} \right\} u_j^2 b_k(t_j - t_i) \tag{10}$$

To obtain a correlation function $R(\tau_k)$ and finally a power spectral density, the correlation coefficient function $\rho(\tau_k)$ is expanded by an estimate of the velocities variance $\sigma_u^2$ yielding

$$R(\tau_k) = \sigma_u^2 \rho(\tau_k) \tag{11}$$

The velocity variance is estimated as

$$\sigma_u^2 = \frac{\sum\limits_{i=0}^{N-1} w_i u_i^2}{\sum\limits_{i=0}^{N-1} w_i} + \sigma_{\bar{u}}^2. \tag{12}$$

The weights $w_i$ are as before for the mean estimate and $\sigma_{\bar{u}}^2$ again is the estimate of the estimation variance of the mean estimator. Note that in the variance estimate the data are assumed to be mean-free in both cases (naturally or by estimating and removing the mean from the data), therefore $u_i$ is given in both cases instead of $u_i - \bar{u}$.

The final spectral estimate is again obtained from the correlation by a discrete Fourier transform (DFT) as given above.

## 3.3 Fuzzy Slotting

Fuzzy slotting [26, 18] is known as a means to further decrease the estimation variance especially at high frequencies. Instead of sharp boundaries between the slots of inter-arrival times between all pairs of data samples, given by the slot window function (of the $k^{\text{th}}$ slot) $b_k(\Delta t)$, smoother, triangular and overlapping window functions

$$b_k(\Delta t) = \left\{ \begin{array}{ll} 1 - \left| \frac{\Delta t}{\Delta \tau} - k \right| & \text{for } |\Delta t - k \Delta \tau| < \Delta \tau \\ 0 & \text{otherwise} \end{array} \right. \tag{13}$$

are used (Fig. 1).

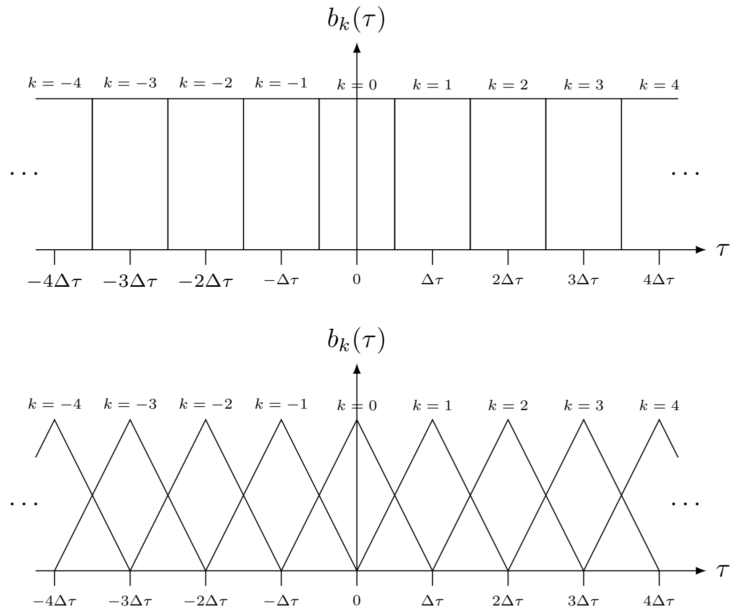


Figure 1: Sharp slotting vs. fuzzy slotting of inter-arrival times

### 3.4 Estimation of the variance of the mean estimator

Following the derivations in [15] to obtain an estimate of the variance of the mean estimator from the data set, the previous sums $R_u(\tau_k)$ and $R_w(\tau_k)$ can be re-used, yielding

$$\sigma_{\bar{u}}^2 = \frac{\sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R_u(\tau_k) + \sum\limits_{i=0}^{N-1} w_{\mathrm{bw},i} w_{\mathrm{fw},i} u_i^2}{W - \sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R_w(\tau_k)}. \tag{14}$$

with

$$\begin{aligned} W &= \sum_{\substack{i=0 \\ i \neq j}}^{N-1} \sum_{j=0}^{N-1} \left\{ \begin{array}{ll} w_{\mathrm{bw},i} w_{\mathrm{fw},j} & \text{if } t_i < t_j \\ w_{\mathrm{fw},i} w_{\mathrm{bw},j} & \text{if } t_i > t_j \end{array} \right\} & (15) \\ &= 2 \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} w_{\mathrm{bw},i} w_{\mathrm{fw},j} & (16) \\ &= 2 \sum_{j=1}^{N-1} w_{\mathrm{fw},j} \left( \sum_{i=0}^{j-1} w_{\mathrm{bw},i} \right) & (17) \end{aligned}$$

Note, that this equation has been modified compared to [15].

### 3.5 Remarks

A statistical bias due to the correlation of the velocity and the instantaneous data rate are suppressed due to the implementation of the weighting schemes [3, 13, 14, 15]. Since the self-products have been removed from the sums, noise components in the data will not cause systematic errors in the derived statistical functions. An example program can be found at [1].

## References


[1] URL `http://ldvproc.nambis.de/programs/pyLDV.html`.

[2] R J Adrian and C S Yao. Power spectra of fluid velocities measured by laser Doppler velocimetry. *Exp. in Fluids*, 5:17–28, 1987.

[3] P Buchhave, W K George, Jr, and J L Lumley. The measurement of turbulence with the laser Doppler anemometer. In *Annual Review of Fluid Mechanics*, volume 11, pages 443–503. Annual Reviews, Inc., Palo Alto, CA, 1979.

[4] P Buchhave and C M Velte. Reduction of noise and bias in randomly sampled power spectra. *Experiments in Fluids*, 56:79, 2015.

[5] M Gaster and J B Roberts. Spectral analysis of randomly sampled signals. *J. Inst. Maths. Applics.*, 15:195–216, 1975.

[6] M Gaster and J B Roberts. The spectral analysis of randomly sampled records by a direct transform. *Proc. R. Soc. Lond. A.*, 354:27–58, 1977.

[7] W K George. Quantitative measurement with the burst-mode laser Doppler anemometer. *Exp. Therm. and Fluid Sci.*, 1:29–40, 1988.

[8] W K George, P D Beuther, and J L Lumley. Processing of random signals. pages 757–800, 1978. Proceedings of the dynamic flow conference, Skovlunde, Denmark, (PO Box 121, DK-2740 Skovlunde).

[9] W T Mayo, Jr. A discussion of the limitations and extensions of power spectrum estimation with burst counter LDV systems. pages 90–104, 1974. Proceedings of the second international workshop on laser velocimetry, (Bull. No. 144), Purdue Univ., West Lafayette, Ind.

[10] W T Mayo, Jr, M T Shay, and S Ritter. The development of new digital data processing techniques for turbulence measurements with a laser velocimeter, 1974.

[11] S Moreau, G Plantier, J-C Valière, H Bailliet, and L Simon. Estimation of power spectral density from laser Doppler data via linear interpolation and deconvolution. *Experiments in Fluids*, 50:179–188, 2011.

[12] H Nobach. *Verarbeitung stochastisch abgetasteter Signale: Anwendung in der Laser-Doppler-Anemometrie*. Shaker, Aachen, 1998. ISBN 3-8265-3332-1, Zugl.: Rostock, Univ., Diss., 1997.

[13] H Nobach. Processing of stochastic sampled data in laser Doppler anemometry. In *Proc. 3rd International Workshop on Sampling Theory and Applications, Loen, Norway*, pages 149–154, 1999.

[14] H Nobach. Local time estimation for the slotted correlation function of randomly sampled LDA data. *Exp. in Fluids*, 32:337–345, 2002.

[15] H Nobach. Corrections to the direct spectral estimation for laser Doppler data. *Exp. in Fluids*, 56:109, 2015.

[16] H Nobach. Fuzzy time quantization and local normalization for the direct spectral estimation from laser Doppler velocimetry data. *Exp. in Fluids*, 56:182, 2015.

[17] H Nobach. A model-free noise removal for the interpolation method of correlation and spectral estimation from laser Doppler data. *Exp. in Fluids*, 56:100, 2015.

[18] H Nobach, E Müller, and C Tropea. Correlation estimator for two-channel, non-coincidence laser-Doppler-anemometer. In *Proc. 9th Int. Symp. on Appl. of Laser Techn. to Fluid Mechanics*, Lisbon, Portugal, 1998. paper 32.1.

[19] H Nobach, E Müller, and C Tropea. Efficient estimation of power spectral density from laser Doppler anemometer data. *Experiments in Fluids*, 24:499–509, 1998.

[20] J B Roberts and D B S Ajmani. Spectral analysis of randomly sampled signals using a correlation-based slotting technique. *IEE Proc.*, 133(2):153–162, 1986.

[21] P F Scott. Random sampling theory and its application to laser velocimeter turbulent spectral measurements. Technical report, General Electric Co., Corporate Res. and Development, 1974. Report No. 74CRD216, Tech, Info Series.

[22] P F Scott. *Distortion and Estimation of the Autocorrelation Function and Spectrum of a Randomly Sampled Signal*. PhD thesis, Rensselear Polytechnic Institute, Troy, NY, 1976.

[23] M T Shay. *Digital Estimation of Autocovariance Functions and Power Spectra from Randomly Sampled Data Using a Lag Product Technique*. PhD thesis, Texas A&M University, College Station, TX, 1976.

[24] L Simon and J Fitzpatrick. An improved sample-and-hold reconstruction procedure for estimation of power spectra from LDA data. *Exp. in Fluids*, 37:272–280, 2004.

[25] M J Tummers, D M Passchier, and P A Aswatha Narayana. LDA measurements of time- and spatial correlation functions in an adverse pressure gradient wake. In *Proc. ASME/JSME Fluids Eng. and Laser Anemometry Conf.*, pages 347–355, Hilton Head Island, South Carolina, USA, 1995. FED-Vol. 229.

[26] H R E van Maanen, H Nobach, and L H Benedict. Improved estimator for the slotted autocorrelation function of randomly sampled LDA data. *Meas. Sci. Technol.*, 10(1):L4–L7, 1999.

[27] H R E van Maanen and M J Tummers. Estimation of the autocorrelation function of turbulent velocity fluctuations using the slotting technique with local normalization. In *Proc. 8th Int. Symp. on Appl. of Laser Techn. to Fluid Mechanics*, Lisbon, Portugal, 1996. paper 36.4.

[28] C M Velte, W K George, and P Buchhave. Estimation of burst-mode LDA power spectra. *Exp. in Fluids*, 55:1674, 2014.