A time-quantization based estimator for autocorrelation and spectral density estimation from laser Doppler anemometry data including local normalization

Holger Nobach

December 3, 2016

Abstract

The time-quantization based estimation for calculating the autocorrelation function and the spectrum from laser Doppler data is revisited and extended by recently developed processing steps.

1 Introduction

For calculating the correlation function or the power spectral density from randomly sampled data from laser Doppler velocity measurements, estimation procedures, which consider the specific characteristics of LDV data are required, namely the sampling of the flow velocity at random arrival times, the data noise and the correlation of the sampling rate and the instantaneous velocity. Much effort has been put onto autocorrelation and autospectral estimators following three different estimator classes, slot correlation, estimating a correlation function (correlogram) from the data [3, 26, 27, 9, 10, 13, 14, 18, 20, 21, 22, 23, 25], direct spectral estimators, estimating a spectrum (periodogram) directly from the randomly sampled data [3, 4, 5, 6, 7, 8, 15, 16, 28] and interpolation methods of the randomly sampled LDV data set yielding a continuous velocity over time, which then is re-sampled equidistantly with a given sampling frequency and processed by usual signal processing tools for equidistantly sampled data, including corrections of systematic errors [2, 11, 19, 24] and noise removal [17, 19]. A fourth processing method, using quantitized arrival times, has not achieved much attention so far. It has been used before, in [12], however, it is broadly available only after publication of [4], where it has been used to accelerate the direct estimation of auto-spectra in combination with a normalization of the appropriate correlation function.

2 The data set

In the following processing steps a sets of irregularly sampled velocity data $u_i = u(t_i)$ at sampling times $t_i, i = 0 \dots N-1$ is assumed together with individual weights w_i according to the velocity smaples u_i, e.g. the particle's transit times. If individual weights are not available, the inter-arrival times can be used for weighting, where both, the forward and the backward inter-arrival times are necessary for the correlation and spectral estimations.

$$\begin{aligned} w_{\mathrm{bw},i} &= t_i - t_{i-1} \\ w_{\mathrm{fw},i} &= t_{i+1} - t_i \end{aligned}$$

To avoid that gaps in the data stream of experimental data lead to improperly large weights, as has been observed in experiments, all inter-arrival time weights derived from inter-arrival times larger than five times the mean inter-arrival time are set to zero. Due to this one looses only about 0.7 % of useful data, while the outliers of large inter-arrival times are suppressed effectively.

3 Time-quantization based Estimator

The main idea of the time-quantized estimator is to quantize the arrival times before the statistical analysis. The time quantization generates a quasi-equidistant data set with gaps, where no data samples are available. If the gaps are filled with zeros, this finally yields a real equidistant sampled data set.[1] The calculation of the autocorrelation function or the auto-spectrum can be performed by tools known from equidistantly sampled data, namely either in the time domain (correlogram) or in the frequency domain (periodogram). Their results are equivalent in that sense, that they are related to each other by the Wiener-Khinchin theorem. Since both ways are possible, the calculation in the spectral domain is followed here, since this may significantly reduce the numerical effort by appying the fast Fourier transform.

If forward-backward inter-arrival time weighting is used, this fast method cannot be applied, because different weighting factors must be applied to the individual velocity values depending on the temporal sequence of any two velocity samples multiplied. In this case, the calculation is performed like the direct spectral estimation with the quantized arrival times (see [1]).

Only if individual weighting is used (e.g. transit-time weighting), then the weighting factors are different for the individual velocity samples, but stay unchanged for any second velocity sample. In this case the fast Fourier transforms can be applied to the products of the weighting factors and the velocity samples. Local normalization [27, 25] is adapted to the time-quantization based estimation method and introduced as optional extensions of the base algorithm. The Python code of the estimator is available at [1].

[1] This data set is comparable to a data set obtained by the interpolation and re-sampling method, except for the fact that between the original data samples, there are zeros instead of the interpolated values.

3.1 Base Algorithm

The following descriptions are valid only for the individual weighting (e.g. transit-time weighting). If forward-backward inter-arrival time weighting is used, instead the direct spectral estimation processing is used with quantized arrival times (see [1]). In both cases, the originally irregularly sampled data set is transformed into a quasi-equidistant data set by quantizing the arrival times. The gaps between original samples are filled with zeros. This is done for both, the weighted velocity samples as well as for the weighting factors themselves. For these equidistantly sampled data sets, the spectra and finally the correlation functions are calculated. Then the correlation function of the weighted velocity samples is normalized by the correlation function of the weights to correct e.g. for processor dead times. Finally, Bessel's correction is applied following [15] and the final spectrum is obtained by a Fourier transform.

1. derivation of the equidistantly sampled data sets

   Taking the fundamental time resolution of the derived correlation function to be $\Delta\tau = {}^{1}/_{F}$ and F the fundamental frequency, three equidistant data series are derived, namely $s'_{0,i} = s'_0(t_i) = s'_0(i\Delta\tau)$, giving the sum of all weights within each interval $t_{i-1} \leq t < t_i$, $s'_{1,i} = s'_1(t_i) = s'_1(i\Delta\tau)$ giving the sum of all weighted velocities within this interval and, for local normalization, $s'_{2,i} = s'_2(t_i) = s'_2(i\Delta\tau)$ giving the sum of all products of the weights and the appropriate velocities squared within this interval.

2. calculation of the primary spectra

$$S_u(f_j) = \frac{T_\mathrm{B}\left[U'^*(f_j)U'(f_j) - \sum\limits_{i=0}^{N-1} w_i^2 u_i^2\right]}{W'^*(0)W'(0) - \sum\limits_{i=0}^{N-1} w_i^2} \tag{1}$$

   and

$$S_w(f_j) = \frac{T_\mathrm{B}\left[W'^*(f_j)W'(f_j) - \sum\limits_{i=0}^{N-1} w_i^2\right]}{W'^*(0)W'(0) - \sum\limits_{i=0}^{N-1} w_i^2} \tag{2}$$

   with

$$U'(f_j) = \mathrm{DFT}\{s'_{1,i}\} = \sum_{i=0}^{J-1} s'_{1,i} \mathrm{e}^{-2\pi \mathrm{i} f_j t_i} \tag{3}$$

   and

$$W'(f_j) = \mathrm{DFT}\{s'_{0,i}\} = \sum_{i=0}^{J-1} s'_{0,i} \mathrm{e}^{-2\pi \mathrm{i} f_j t_i} \tag{4}$$

   by means of the discrete Fourier transform DFT. The primary spectra are calculated for J different frequencies $f_j = j\Delta f, j = -\lfloor {}^{J}/_{2} \rfloor \ldots \lfloor {}^{(J-1)}/_{2} \rfloor$ with $J = [2T_\mathrm{B}F]$ and $\Delta f = {}^{F}/_{J} = {}^{1}/_{2T_\mathrm{B}}$.

3. transformation into correlation functions and limiting the correlation function (alternative to block averaging)

$$R_u(\tau_k) = F \cdot \mathrm{IDFT}\left\{S_u(f_j)\right\} = \frac{1}{J\Delta\tau} \sum_{j=-\lfloor J/2 \rfloor}^{\lfloor (J-1)/2 \rfloor} S_u(f_j) \mathrm{e}^{2\pi \mathrm{i} f_j \tau_k} \tag{5}$$

and

$$R_w(\tau_k) = F \cdot \mathrm{IDFT}\left\{S_w(f_j)\right\} = \frac{1}{J\Delta\tau} \sum_{j=-\lfloor J/2 \rfloor}^{\lfloor (J-1)/2 \rfloor} S_w(f_j) \mathrm{e}^{2\pi \mathrm{i} f_j \tau_k} \tag{6}$$

by means of the inverse discrete Fourier transform IDFT. The correlation function is calculated for K different time lags, where K is chosen significantly smaller than J according to the considered correlation interval $[-T_\mathrm{C}/2 : T_\mathrm{C}/2), T_\mathrm{C} = K/F$, for $\tau_k = k\Delta\tau, k = -\lfloor K/2 \rfloor \dots \lfloor (K-1)/2 \rfloor$,

4. normalization (correction of dead time influences and normalization to an estimate of the correlation of the original signal) and Bessel's correction

$$R(\tau_k) = \frac{R_u(\tau_k)}{R_w(\tau_k)} + \sigma_{\bar{u}}^2, \tag{7}$$

for $\tau_k = k\Delta\tau, k = -\lfloor K/2 \rfloor \dots \lfloor (K-1)/2 \rfloor$ with $\sigma_{\bar{u}}^2$ an estimate of the variance of the mean estimation. An appropriate estimator follows.

5. back-transformation into the final spectrum by means of the discrete Fourier transform DFT

$$S(f_j) = \Delta\tau \cdot \mathrm{DFT}\left\{R(\tau_k)\right\} = \Delta\tau \sum_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R(\tau_k) \mathrm{e}^{-2\pi \mathrm{i} f_j \tau_k} \tag{8}$$

with the (reduced) spectral resolution $\Delta f = F/K = 1/T_\mathrm{C}$ for $f_j = j\Delta f, j = -\lfloor K/2 \rfloor \dots \lfloor (K-1)/2 \rfloor$

3.2 Local Normalization

A further reduction of random errors can be achieved with the local normalization [26, 27, 25]. The main idea of this correction is to normalize the correlation estimate at every time lag by a variance estimate, which corresponds to exactly the data samples used for the correlation estimate. The result is a correlation coefficient ρ in the range $[-1 : 1]$. The method has been developed originally for the slotting technique. However, it can be adapted also to the direct spectral estimator as one step of the various corrections acting on the correlation function. The new estimate of the correlation coefficient then becomes

$$\rho(\tau_k) = \frac{R_u(\tau_k) + \sigma_{\bar{u}}^2 R_w(\tau_k)}{\sqrt{\left[R_{u_1}(\tau_k) + \sigma_{\bar{u}}^2 R_w(\tau_k)\right]\left[R_{u_2}(\tau_k) + \sigma_{\bar{u}}^2 R_w(\tau_k)\right]}}, \tag{9}$$

with $f_j = j\Delta f, j = -\lfloor J/2 \rfloor \ldots \lfloor (J-1)/2 \rfloor$ and $\tau_k = k\Delta\tau, k = -\lfloor K/2 \rfloor \ldots \lfloor (K-1)/2 \rfloor$ (The inverse discrete Fourier transform originally yields $\tau_k = k\Delta\tau, k = -\lfloor J/2 \rfloor \ldots \lfloor (J-1)/2 \rfloor$ where only the range $k = -\lfloor K/2 \rfloor \ldots \lfloor (K-1)/2 \rfloor$ is processed further.) with $R_u(\tau_k)$ and $R_w(\tau_k)$ as above and

$$R_{u_1}(\tau_k) = F \cdot \mathrm{IDFT}\left\{S_{u_1}(f_j)\right\} = \frac{1}{J\Delta\tau} \sum_{j=-\lfloor J/2 \rfloor}^{\lfloor (J-1)/2 \rfloor} S_{u_1}(f_j) \mathrm{e}^{2\pi \mathrm{i} f_j \tau_k} \tag{10}$$

and

$$R_{u_2}(\tau_k) = F \cdot \mathrm{IDFT}\left\{S_{u_2}(f_j)\right\} = \frac{1}{J\Delta\tau} \sum_{j=-\lfloor J/2 \rfloor}^{\lfloor (J-1)/2 \rfloor} S_{u_2}(f_j) \mathrm{e}^{2\pi \mathrm{i} f_j \tau_k} \tag{11}$$

with

$$S_{u_1}(f_j) = \frac{T_\mathrm{B}\left[U_2'^*(f_j) W'(f_j) - \sum\limits_{i=0}^{N-1} w_i^2 u_i^2\right]}{W'^*(0) W'(0) - \sum\limits_{i=0}^{N-1} w_i^2} \tag{12}$$

and

$$S_{u_2}(f_j) = \frac{T_\mathrm{B}\left[W'^*(f_j) U_2'(f_j) - \sum\limits_{i=0}^{N-1} w_i^2 u_i^2\right]}{W'^*(0) W'(0) - \sum\limits_{i=0}^{N-1} w_i^2} \tag{13}$$

with

$$U_2'(f) = \mathrm{DFT}\{s_{2,i}'\} = \sum_{i=0}^{J-1} s_{2,i}' \mathrm{e}^{-2\pi \mathrm{i} f_j t_i} \tag{14}$$

Note that J must be chosen odd to ensure that $R_{u_1}(\tau_k)$ and $R_{u_2}(\tau_k)$ are real. This is required only for the case of local normalization applied.

To obtain a correlation function $R(\tau_k)$ and finally a power spectral density, the correlation coefficient function $\rho(\tau_k)$ is expanded by an estimate of the velocities variance σ_u^2 yielding

$$R(\tau_k) = \sigma_u^2 \rho(\tau_k) \tag{15}$$

The velocity variance is estimated as

$$\sigma_u^2 = \frac{\sum\limits_{i=0}^{N-1} w_i u_i^2}{\sum\limits_{i=0}^{N-1} w_i} + \sigma_{\bar{u}}^2. \tag{16}$$

The weights w_i are as before for the mean estimate and $\sigma_{\bar{u}}^2$ again is the estimate of the variance of the mean estimator. Note that in the variance estimate the data are assumed to be mean-free in both cases (naturally or by estimating and removing the mean from the data), therefore u_i is given in both cases instead of $u_i - \bar{u}$.

The final spectral estimate is again obtained from the correlation by a discrete Fourier transform (DFT) as given above.

3.3 Estimation of the variance of the mean estimator

Following the derivations in [15] to obtain an estimate of the variance of the mean estimator from the data set, the previous correlation estimates $R_u(\tau_k)$ and $R_w(\tau_k)$ can be re-used, yielding

$$\sigma_{\bar{u}}^2 = \frac{\frac{1}{T_\mathrm{B}F} \sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R_u(\tau_k) + \frac{1}{W'^*(0)W'(0) - \sum\limits_{i=0}^{N-1} w_i^2} \sum\limits_{i=0}^{N-1} w_i^2 u_i^2}{1 - \frac{1}{T_\mathrm{B}F} \sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R_w(\tau_k)} \tag{17}$$

3.4 Remarks

A statistical bias due to the correlation of the velocity and the instantaneous data rate are suppressed due to the implementation of the weighting schemes. Since the self-products have been removed from the sums, noise components in the data will not cause systematic errors in the derived statistical functions. An example program can be found at [1] including forward-backward inter-arrival time weighting.

References

[1] URL `http://ldvproc.nambis.de/programs/pyLDV.html`.

[2] R J Adrian and C S Yao. Power spectra of fluid velocities measured by laser Doppler velocimetry. *Exp. in Fluids*, 5:17–28, 1987.

[3] P Buchhave, W K George, Jr, and J L Lumley. The measurement of turbulence with the laser Doppler anemometer. In *Annual Review of Fluid Mechanics*, volume 11, pages 443–503. Annual Reviews, Inc., Palo Alto, CA, 1979.

[4] P Buchhave and C M Velte. Reduction of noise and bias in randomly sampled power spectra. *Experiments in Fluids*, 56:79, 2015.

[5] M Gaster and J B Roberts. Spectral analysis of randomly sampled signals. *J. Inst. Maths. Applics.*, 15:195–216, 1975.

[6] M Gaster and J B Roberts. The spectral analysis of randomly sampled records by a direct transform. *Proc. R. Soc. Lond. A.*, 354:27–58, 1977.

[7] W K George. Quantitative measurement with the burst-mode laser Doppler anemometer. *Exp. Therm. and Fluid Sci.*, 1:29–40, 1988.

[8] W K George, P D Beuther, and J L Lumley. Processing of random signals. pages 757–800, 1978. Proceedings of the dynamic flow conference, Skovlunde, Denmark, (PO Box 121, DK-2740 Skovlunde).

[9] W T Mayo, Jr. A discussion of the limitations and extensions of power spectrum estimation with burst counter LDV systems. pages 90–104, 1974. Proceedings of the second international workshop on laser velocimetry, (Bull. No. 144), Purdue Univ., West Lafayette, Ind.

[10] W T Mayo, Jr, M T Shay, and S Ritter. The development of new digital data processing techniques for turbulence measurements with a laser velocimeter, 1974.

[11] S Moreau, G Plantier, J-C Valière, H Bailliet, and L Simon. Estimation of power spectral density from laser Doppler data via linear interpolation and deconvolution. *Experiments in Fluids*, 50:179–188, 2011.

[12] H Nobach. *Verarbeitung stochastisch abgetasteter Signale: Anwendung in der Laser-Doppler-Anemometrie.* Shaker, Aachen, 1998. ISBN 3-8265-3332-1, Zugl.: Rostock, Univ., Diss., 1997.

[13] H Nobach. Processing of stochastic sampled data in laser Doppler anemometry. In *Proc. 3rd International Workshop on Sampling Theory and Applications, Loen, Norway*, pages 149–154, 1999.

[14] H Nobach. Local time estimation for the slotted correlation function of randomly sampled LDA data. *Exp. in Fluids*, 32:337–345, 2002.

[15] H Nobach. Corrections to the direct spectral estimation for laser Doppler data. *Exp. in Fluids*, 56:109, 2015.

[16] H Nobach. Fuzzy time quantization and local normalization for the direct spectral estimation from laser Doppler velocimetry data. *Exp. in Fluids*, 56:182, 2015.

[17] H Nobach. A model-free noise removal for the interpolation method of correlation and spectral estimation from laser Doppler data. *Exp. in Fluids*, 56:100, 2015.

[18] H Nobach, E Müller, and C Tropea. Correlation estimator for two-channel, non-coincidence laser-Doppler-anemometer. In *Proc. 9th Int. Symp. on Appl. of Laser Techn. to Fluid Mechanics*, Lisbon, Portugal, 1998. paper 32.1.

[19] H Nobach, E Müller, and C Tropea. Efficient estimation of power spectral density from laser Doppler anemometer data. *Experiments in Fluids*, 24:499–509, 1998.

[20] J B Roberts and D B S Ajmani. Spectral analysis of randomly sampled signals using a correlation-based slotting technique. *IEE Proc.*, 133(2):153–162, 1986.

[21] P F Scott. Random sampling theory and its application to laser velocimeter turbulent spectral measurements. Technical report, General Electric Co., Corporate Res. and Development, 1974. Report No. 74CRD216, Tech, Info Series.

[22] P F Scott. *Distortion and Estimation of the Autocorrelation Function and Spectrum of a Randomly Sampled Signal.* PhD thesis, Rensselear Polytechnic Institute, Troy, NY, 1976.

[23] M T Shay. *Digital Estimation of Autocovariance Functions and Power Spectra from Randomly Sampled Data Using a Lag Product Technique.* PhD thesis, Texas A&M University, College Station, TX, 1976.

[24] L Simon and J Fitzpatrick. An improved sample-and-hold reconstruction procedure for estimation of power spectra from LDA data. *Exp. in Fluids*, 37:272–280, 2004.

[25] M J Tummers, D M Passchier, and P A Aswatha Narayana. LDA measurements of time- and spatial correlation functions in an adverse pressure gradient wake. In *Proc. ASME/JSME Fluids Eng. and Laser Anemometry Conf.*, pages 347–355, Hilton Head Island, South Carolina, USA, 1995. FED-Vol. 229.

[26] H R E van Maanen, H Nobach, and L H Benedict. Improved estimator for the slotted autocorrelation function of randomly sampled LDA data. *Meas. Sci. Technol.*, 10(1):L4–L7, 1999.

[27] H R E van Maanen and M J Tummers. Estimation of the autocorrelation function of turbulent velocity fluctuations using the slotting technique with local normalization. In *Proc. 8th Int. Symp. on Appl. of Laser Techn. to Fluid Mechanics*, Lisbon, Portugal, 1996. paper 36.4.

[28] C M Velte, W K George, and P Buchhave. Estimation of burst-mode LDA power spectra. *Exp. in Fluids*, 55:1674, 2014.