Randomly (Stochastic) Sampled Data

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Discussion

Non-equidistant Sampling ⟺ Random Sampling

Discussion

Non-equidistant Sampling ⟺ Random Sampling

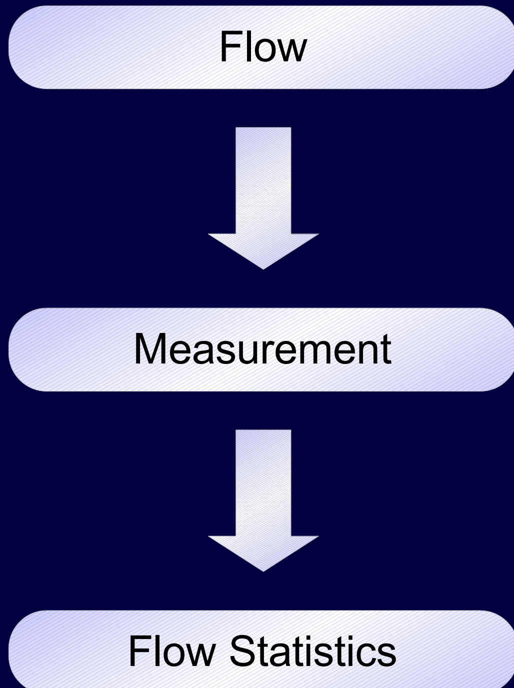
driven ⟺ given

Discussion

Non-equidistant Sampling ⟺ Random Sampling

driven ⟺ given

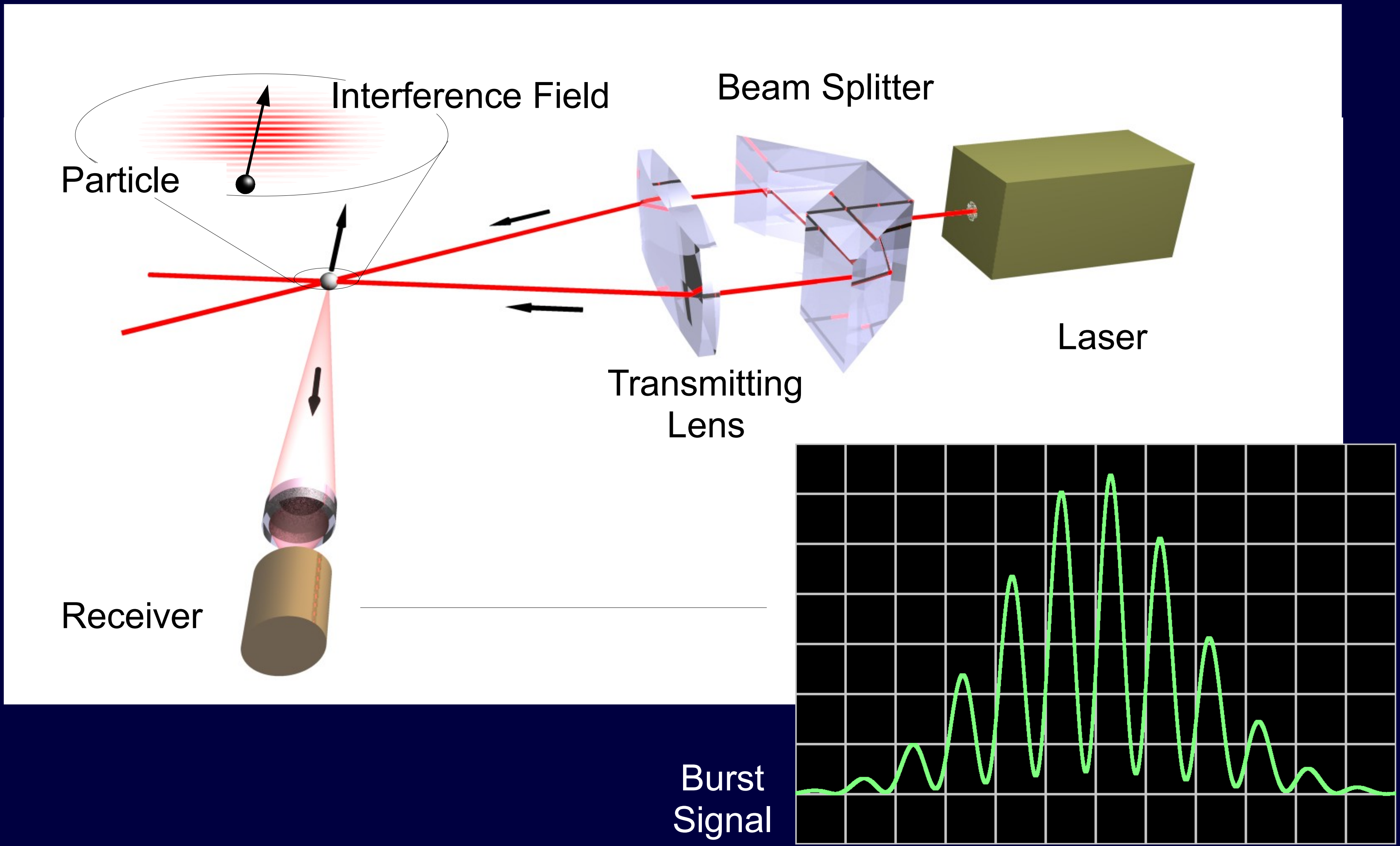
advantage ⟺ disadvantage



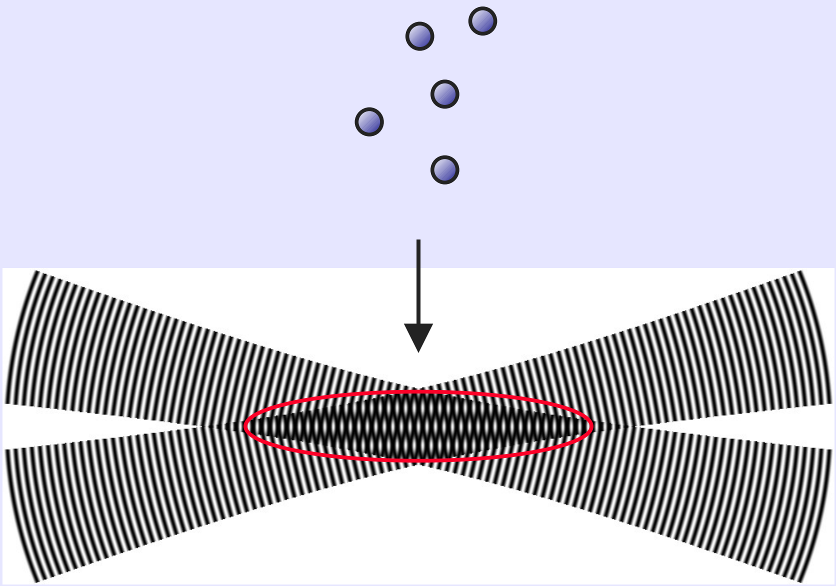
Hot-wire Anemometry
Laser Doppler Anemometry
Particle image velocimetry

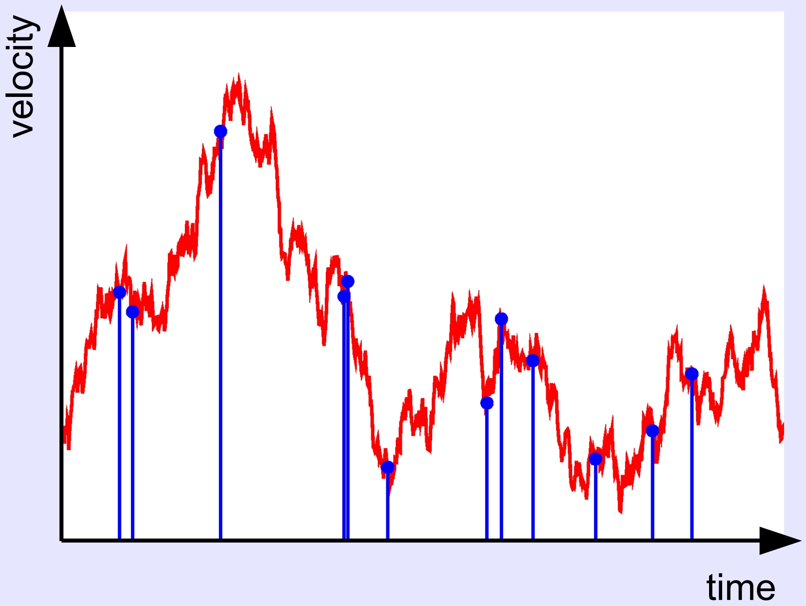
Moments
Probability Density
Cross-/Auto-Correlation Functions
Structure Functions
Power Spectral Density
Temporal / Spatial / Lagrangian

Laser Doppler Anemometry

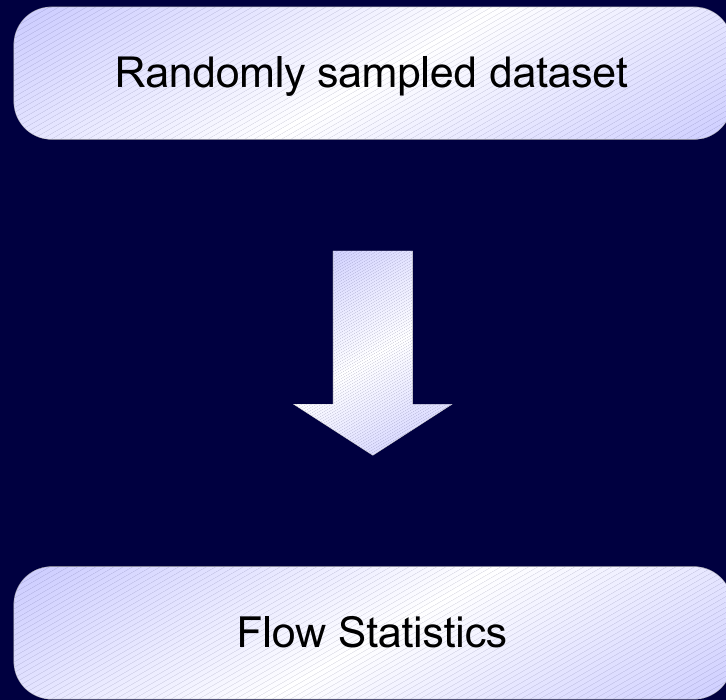


LDA Dataset



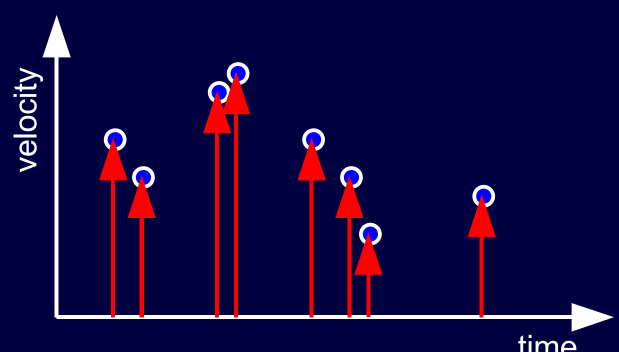


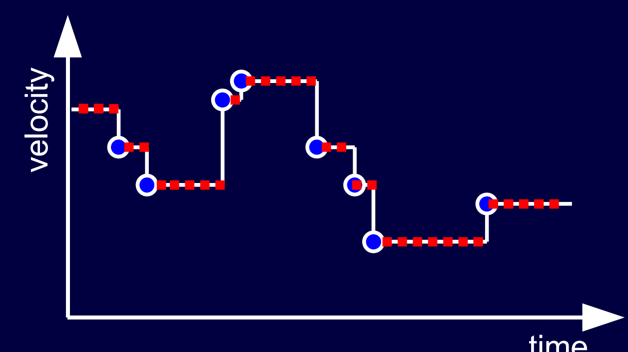
Measurement of single particles	⇨	randomly sampled time series
Uncertainty in frequency measurement	⇨	wide-band noise
Correlation between particle rate and velocity	⇨	statistical bias
Interference of light scattered by different particles	⇨	only single particles allowed

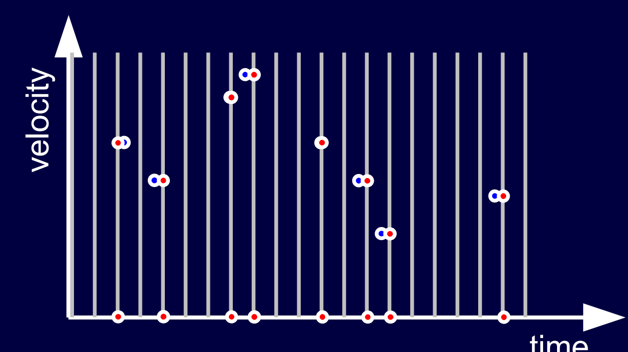

Randomly sampled dataset
Flow Statistics

Randomly sampled dataset

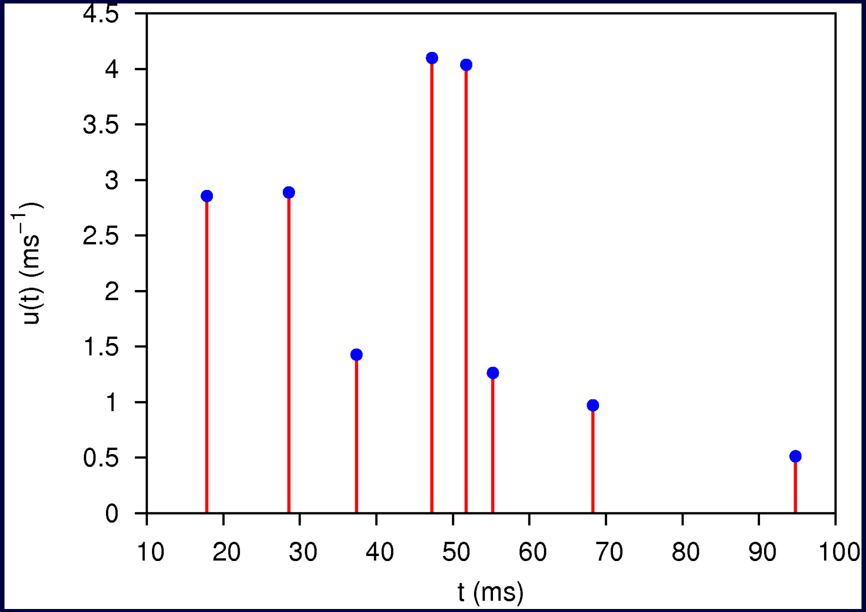
Direct Estimation	Signal Reconstruction and Equidistant re-sampling	Transformation into a sparsely Sampled dataset
◆ Mathematical description (e.g. Series of Diracs) ◆ Including the random sampling ◆ Development of appropriate estimators	◆ Appropriate interpolation/ reconstruction method ◆ Classical signal processing	◆ Quantization of sampling time / time interval ◆ Considering signal gaps ◆ Estimators from process identification

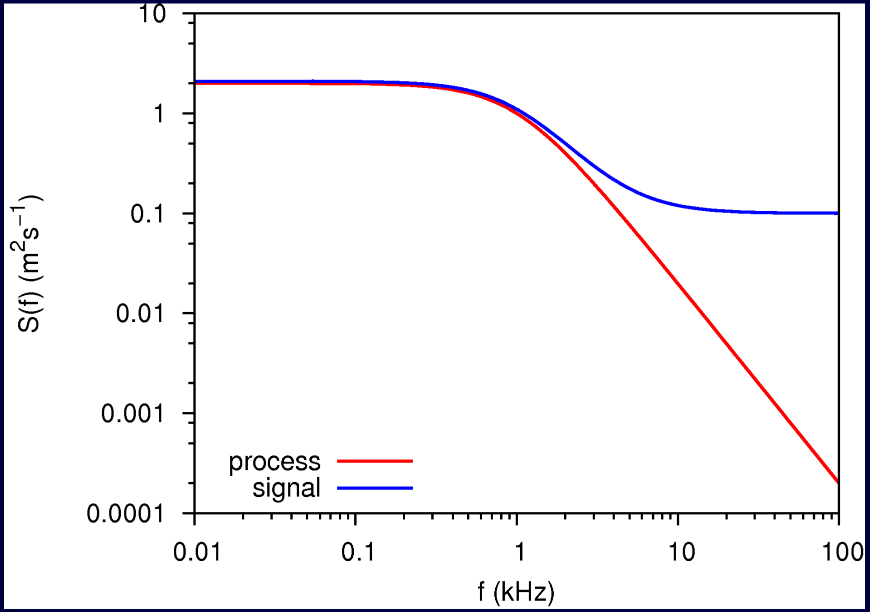






- Principle of direct estimation
- Bias due to random sampling



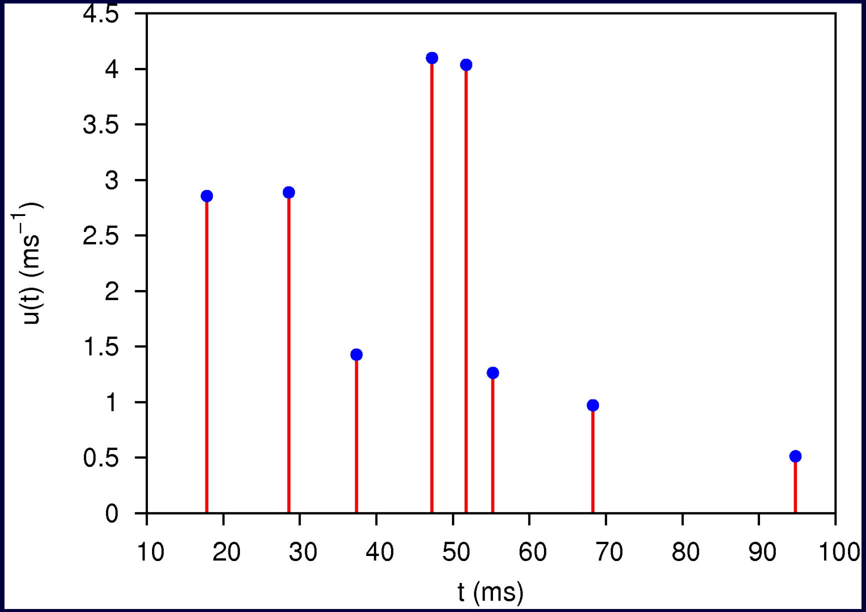


$$S_S(f) \stackrel{\text{def}}{=} \frac{1}{T}\left|\int_0^T u(t)\, e^{-2\pi j f t_i}\right|^2 = \frac{T}{N^2}\left|\sum_{i=1}^{N} u_i e^{-2\pi j f t_i}\right|^2$$

Bias prediction: $E\{S_S\} = S_P + \frac{T}{N}\sigma_u^2$

correction: $\hat{S}_P(f) = \frac{T}{N^2}\left\{\left|\sum_{i=1}^{N} u_i e^{-2\pi j f t_i}\right|^2 - \sum_{i=1}^{N} u_i^2\right\}$

- Principle of direct estimation



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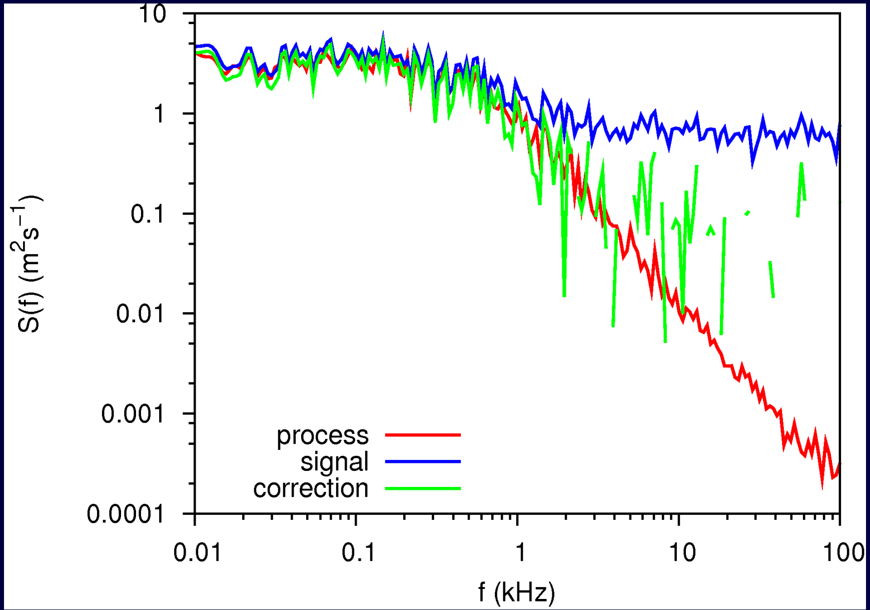
- Bias due to random sampling



Bias prediction: $E\{S_S\} = S_P + \frac{T}{N}\sigma_u^2$

correction: $\hat{S}_P(f) = \frac{T}{N^2}\left\{\left|\sum_{i=1}^{N} u_i e^{-2\pi j f t_i}\right|^2 - \sum_{i=1}^{N} u_i^2\right\}$

LDA dataset

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Reconstruction and Re-sampling

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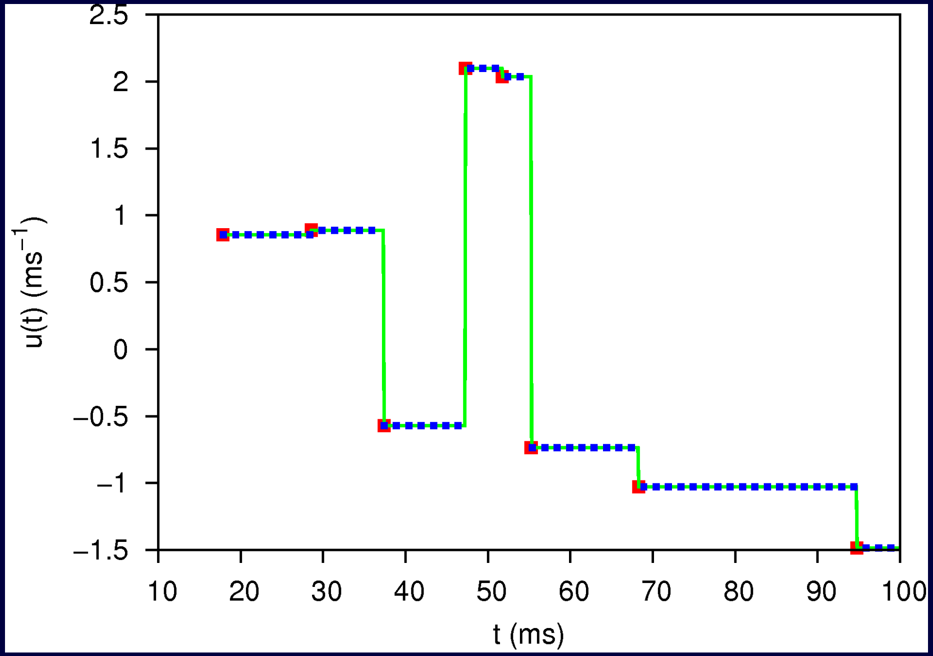
Correlation function

↓

Fourier transform

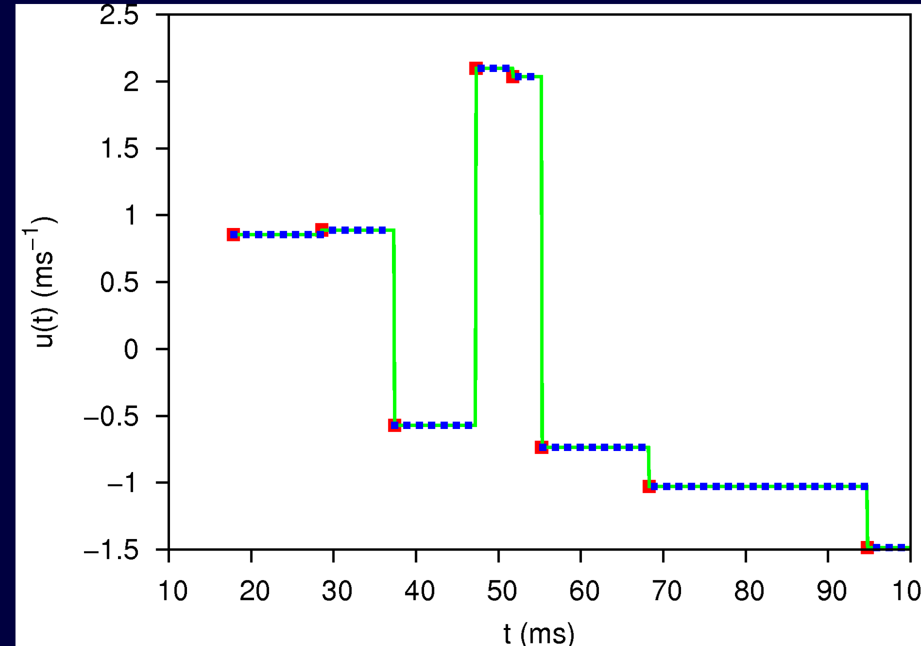
↓

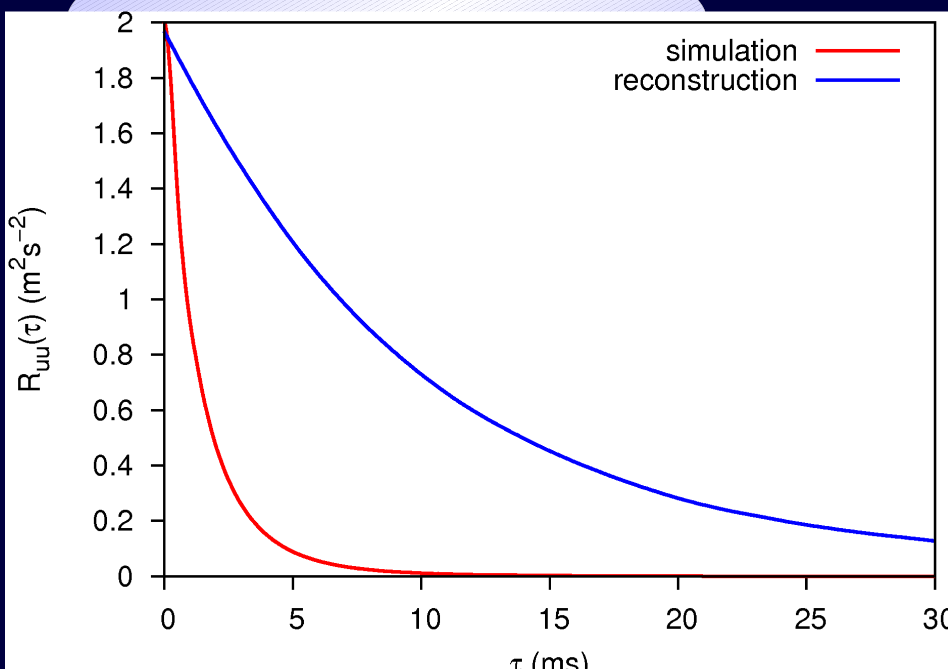
spectrum

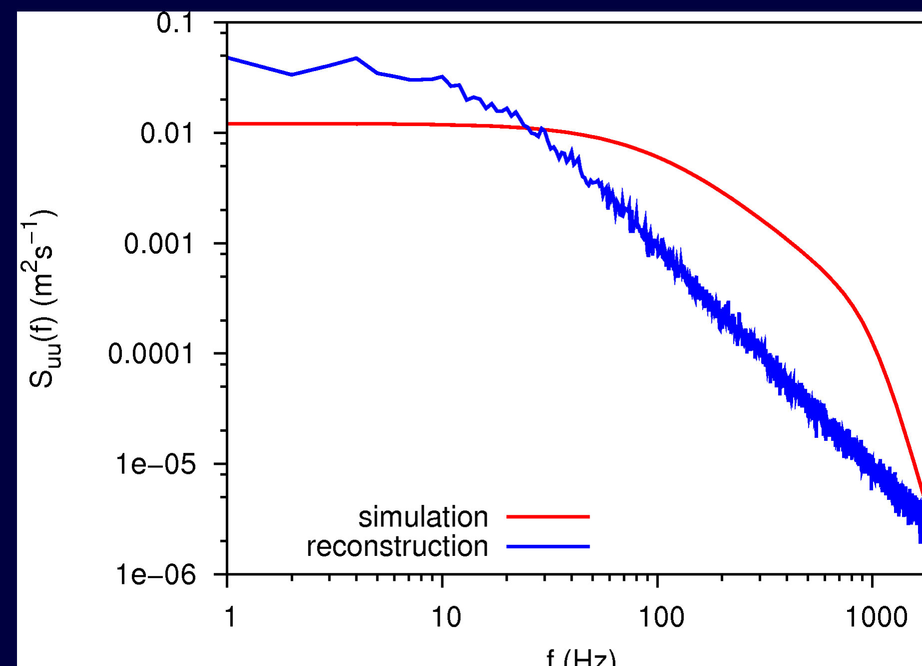


LDA dataset

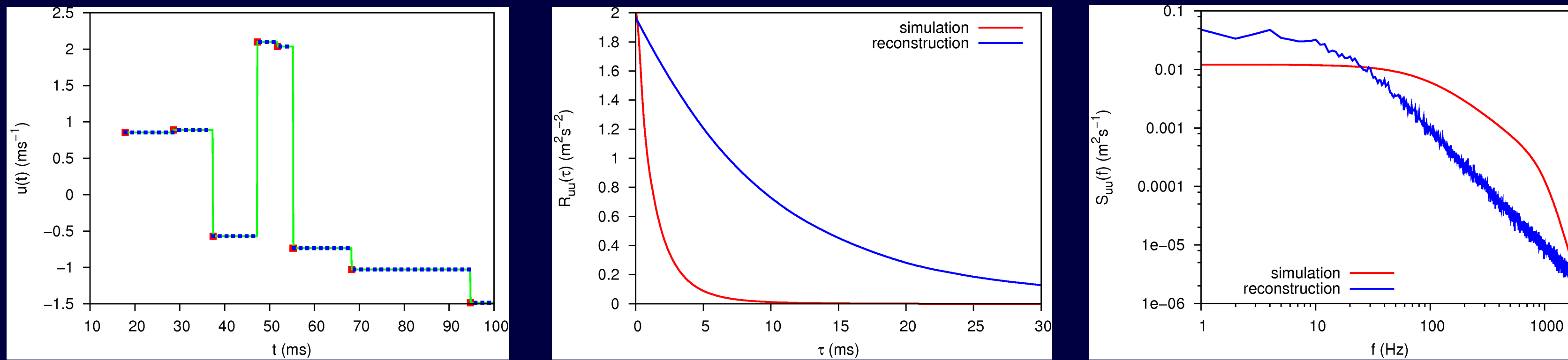
Reconstruction and Re-sampling



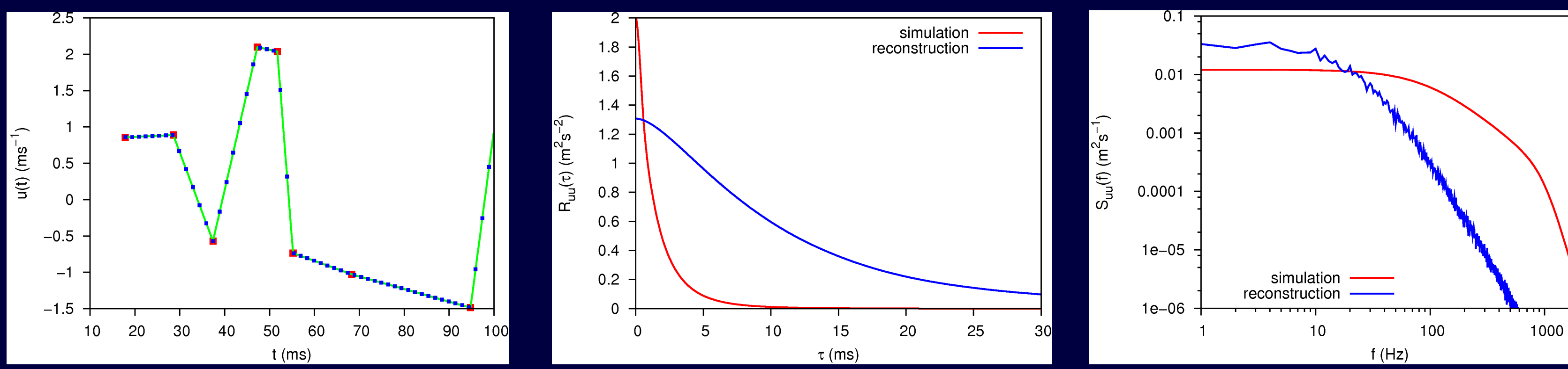




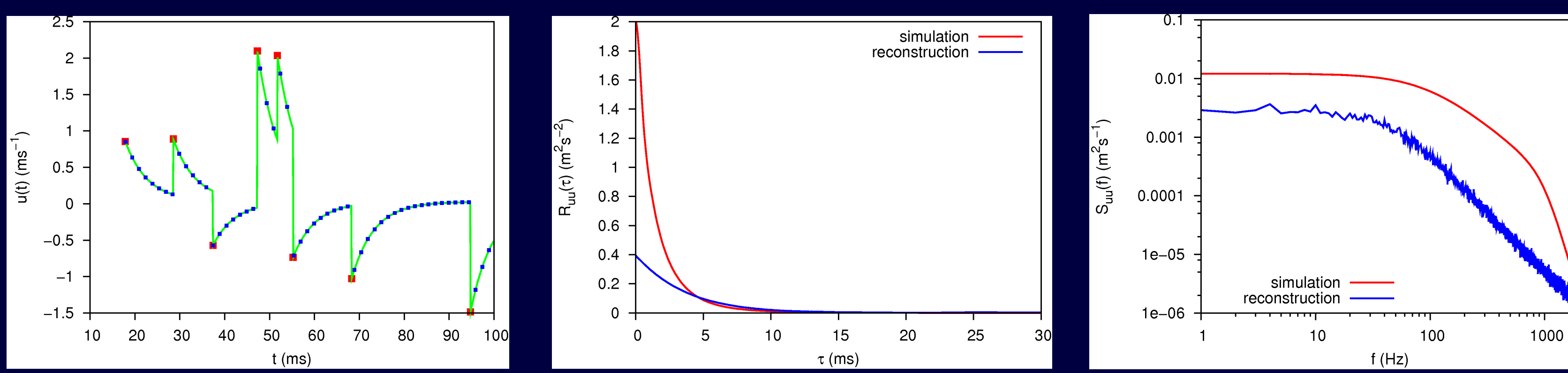
- Sample-and-Hold Reconstruction



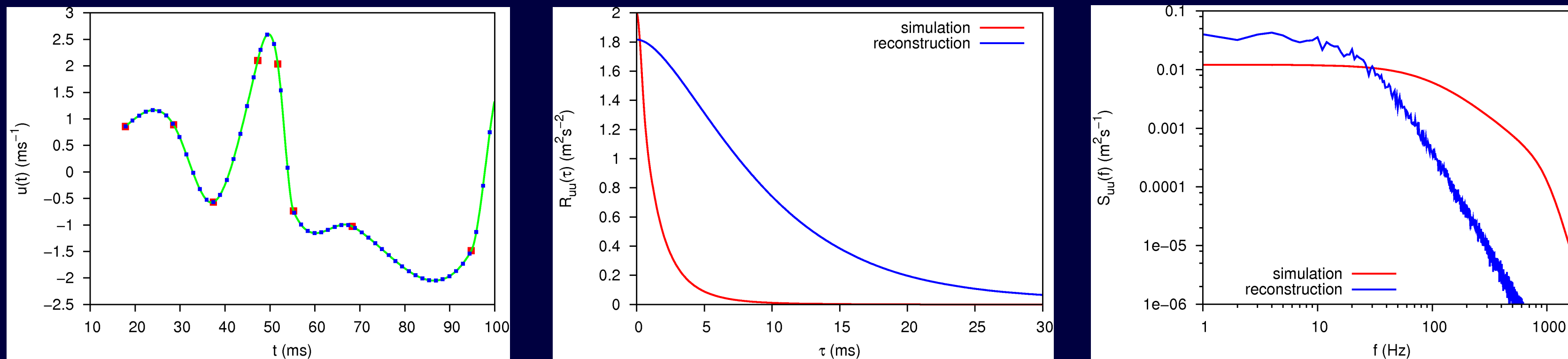
- linear interpolation



- exponential reconstruction



- Spline Interpolation



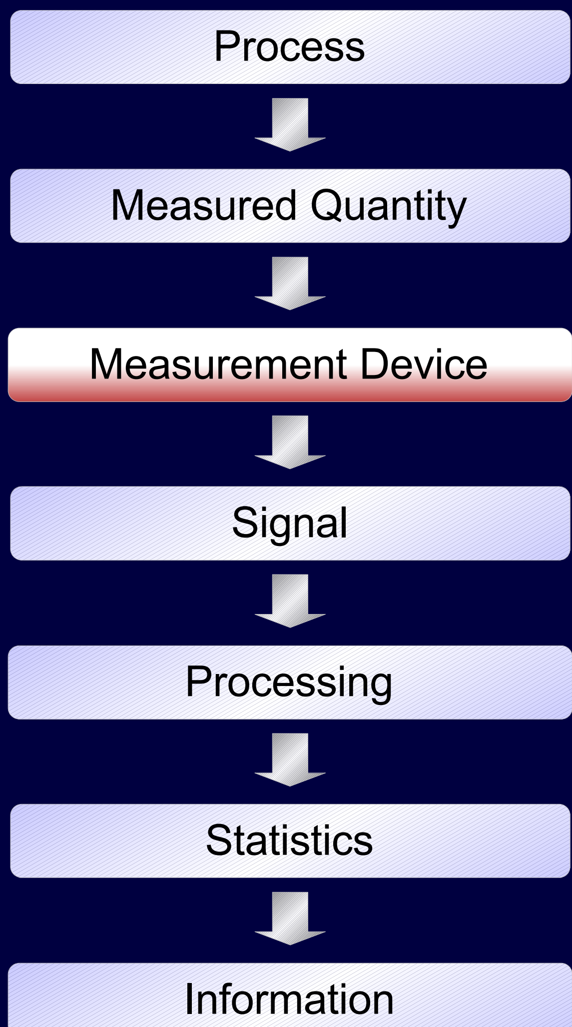
- Kalman Reconstruction
- Shannon Reconstruction
- Projections Onto Convex Sets
- Fractal Reconstruction

For all reconstructions (independent of the reconstruction method):

- At high data rates good reconstruction and conservation of flow statistics.
- At low data rates poor reconstruction and transition to reconstruction statistics (independent of the flow).

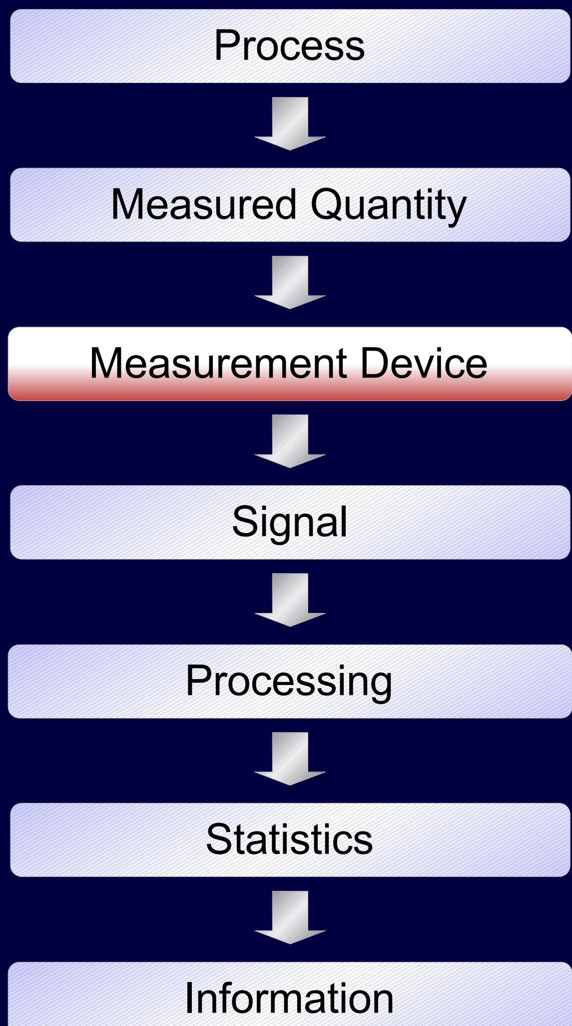
Random Sampling

- Signal Processing

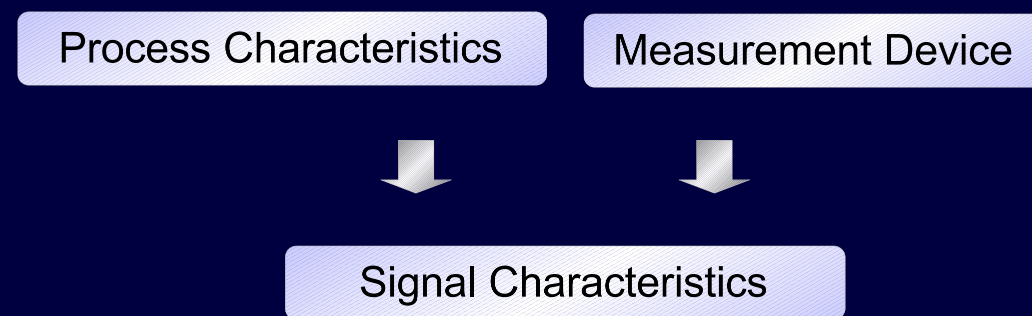


Random Sampling

- Signal Processing



- Measurement Device



1. Analysis of reconstructed dataset

LDA dataset

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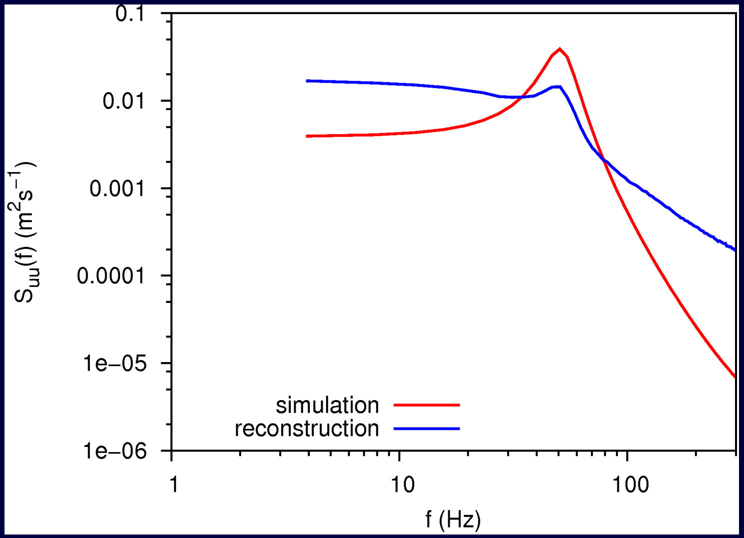
Reconstruction

↓

Resampling

↓

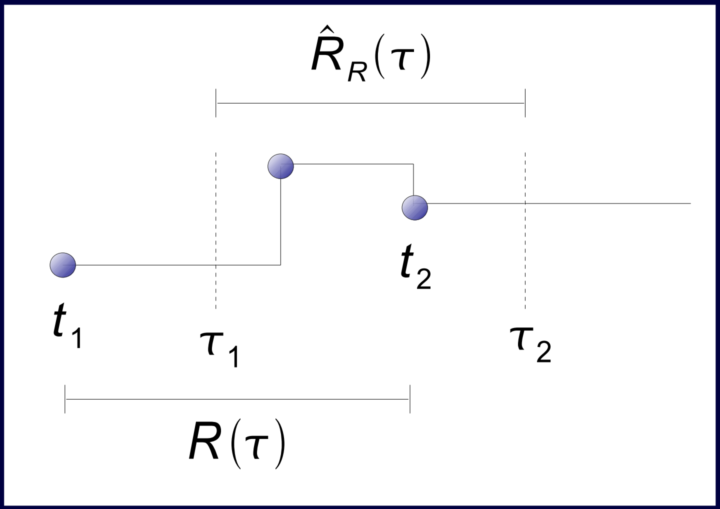
Correlation and Spectral Estimation



2. Filter Estimation

LDA dataset → Reconstruction → Resampling → Correlation and Spectral Estimation

true ACF → $E\{\hat{R}_R(\tau)\} = MR(\tau)$ → expected ACF after reconstruction

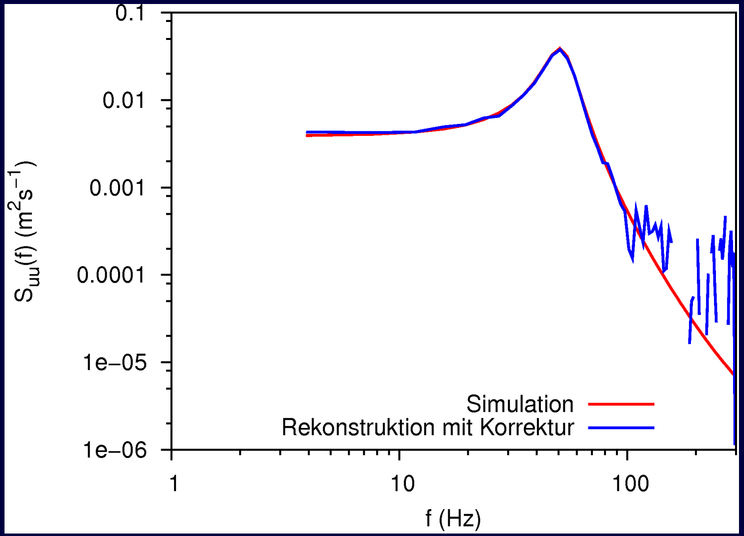


3. Correction

LDA dataset → Reconstruction → Resampling → Correlation and Spectral Estimation

true ACF → $E\{\hat{R}_R(\tau)\} = MR(\tau)$ → expected ACF after reconstruction

empiric ACF after reconstruction → $\hat{R}(\tau) = M^{-1}\hat{R}_R(\tau)$ → corrected ACF



- Sample-and-Hold Reconstruction
  - Reconstruction scheme $u_R(t)=u_i \quad t_i \leqslant t < t_{i+1}$
  - Interpolation filter

$$E\left\{\hat{R}_R(\tau_k)\right\}=e^{-\dot{n}\tau_k}\left\{R(0)+\frac{(e^{\dot{n}\Delta\tau}-1)^2}{1-e^{2\dot{n}\Delta\tau}}\sum_{\xi=1}^{\infty} e^{-\dot{n}\tau_\xi}(1-e^{2\dot{n}\min(k,\xi)\Delta\tau})R(\tau_\xi)\right\}$$

  - Correction

$$\hat{R}(\tau_k)=\begin{cases}\hat{R}_R(0) & \text{for } k=0\\ (2c+1)\hat{R}_R(\tau_k)-c\left[\hat{R}_R(\tau_{k-1})+\hat{R}_R(\tau_{k+1})\right] & \text{otherwise}\end{cases} \qquad c=\frac{e^{-\dot{n}\Delta\tau}}{(1-e^{-\dot{n}\Delta\tau})^2}$$

- Proportional One-Point Reconstruction (exp., Correlation Coefficient, S&H)
  - Reconstruction scheme $u_R(t)=u_i f_R(t-t_i) \quad t_i \leqslant t < t_{i+1}$
  - Interpolation filter

$$E\left\{\hat{R}_R(\tau_k)\right\}=R(0)\sum_{i=-\infty}^{0} f_R(-\tau_i) f_R(\tau_k-\tau_i)\left(1-e^{-\dot{n}\Delta\tau}\right)e^{-\dot{n}(\tau_k-\tau_i)}$$
$$+\sum_{\xi=1}^{\infty} R(\tau_\xi)\sum_{i=1}^{\min(k,\xi)} f_R(\tau_\xi-\tau_i) f_R(\tau_k-\tau_\xi)\left(1-e^{-\dot{n}\Delta\tau}\right)^2 e^{-\dot{n}(\tau_k-2\tau_i+\tau_\xi)}$$

  - Numerical correction by solving a system of linear equations

- Higher-order interpolation schemes
  - Principally possible
  - Numerical effort rapidly increases with number of samples
  - Marginal advantages compared with Sample-and-Hold Interpolation

LDA dataset

↓

S&H reconstruction and Equidistant re-sampling

↓

Correlation function

↓

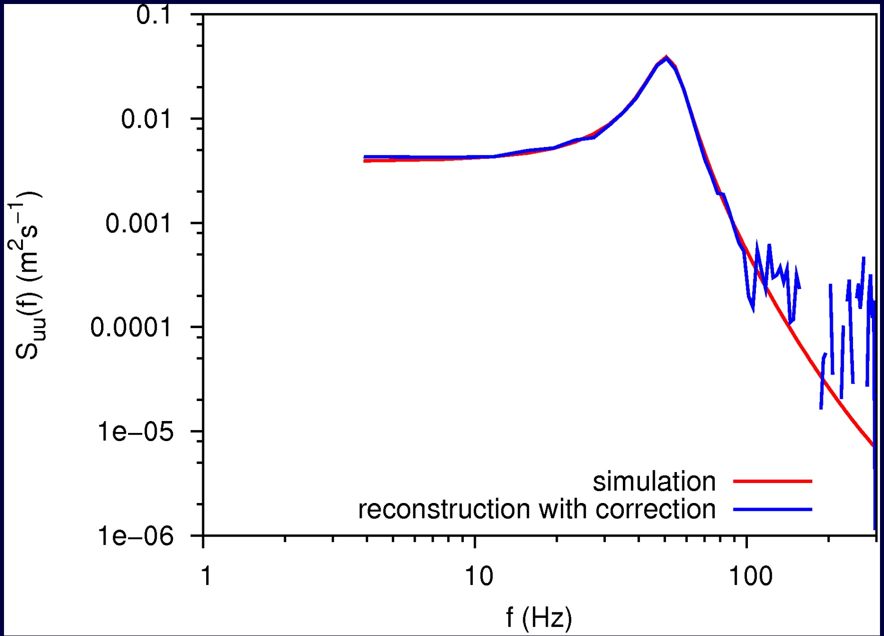
Estimate of the data rate → Filter correction

↓

Fourier transform

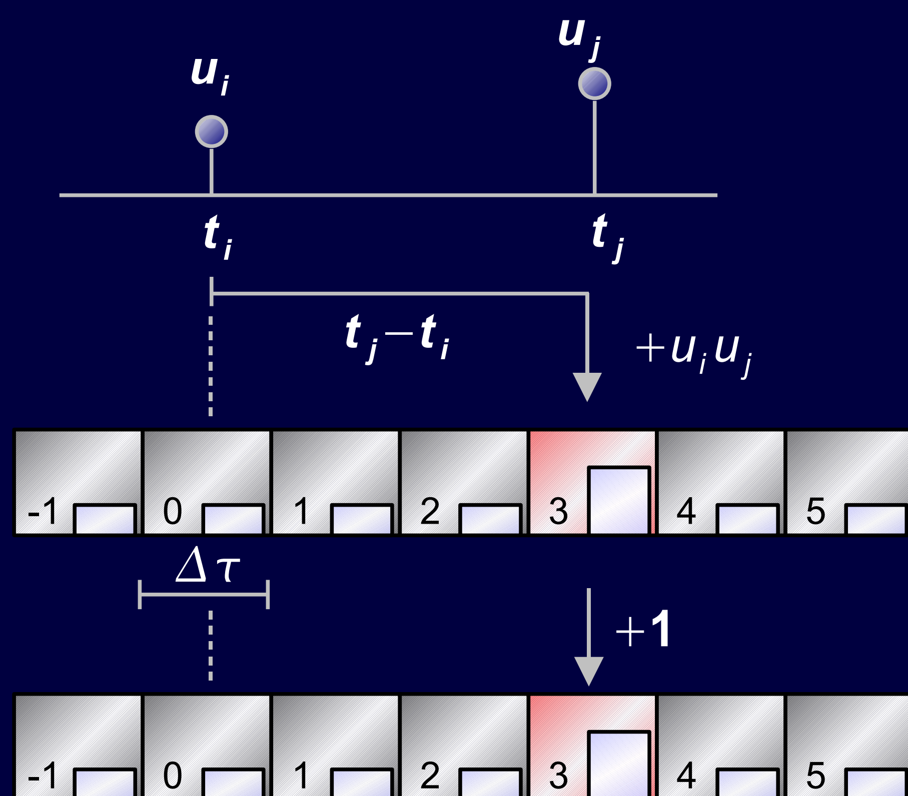
↓

spectrum



Slot Correlation

$$i=1\dots N$$
$$j=1\dots N \qquad j\neq i$$



$$n_k=\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} u_i u_j b_k(t_j-t_i)$$

$$d_k=\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} b_k(t_j-t_i)$$

$$b_k(\Delta t)=\begin{cases}1 & \text{for}\,(k-1/2)\Delta\tau\leq\Delta t<(k+1/2)\Delta\tau\\ 0 & \text{otherwise}\end{cases}$$

ACF
$$\hat{R}_k=n_k/d_k$$

Randomly sampled dataset

S&H reconstruction and equidist. re-sampling

- correction of the data rate filter
- Processing of long datasets by ACF truncation
- Use of fast algorithms (FFT)
- Commercial products

Slot Correlation

- Local Normalization
- Fuzzy Slotting
- Local Time Estimation
- Sample weighting (Transit-Time / Forward-Backward-Arrival-Time)
- Slightly smaller random errors
- Offline processing

model-based noise and variance estimation

Time scales

- Taylor time scale
- Integral time scale

Power density spectrum

- Continuous Fourier Transform
- Variable Windowing

Spatial Correlation Function

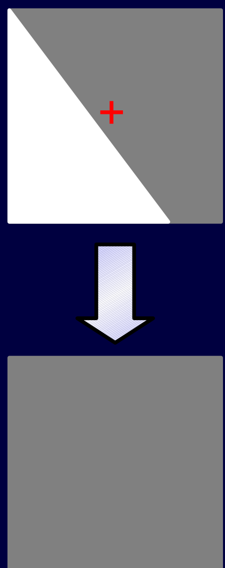
- Time-Space Transforms

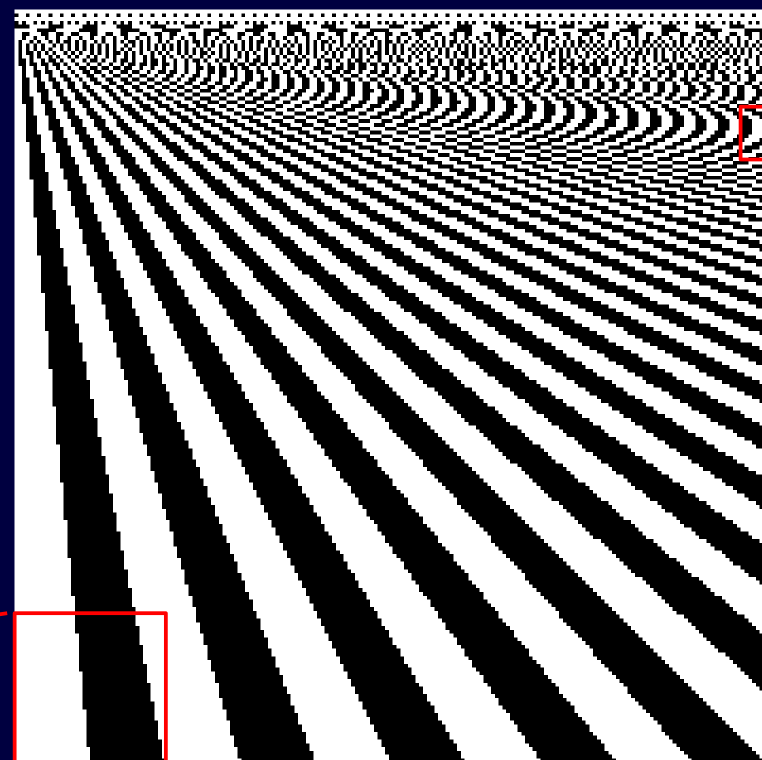
Random Sampling

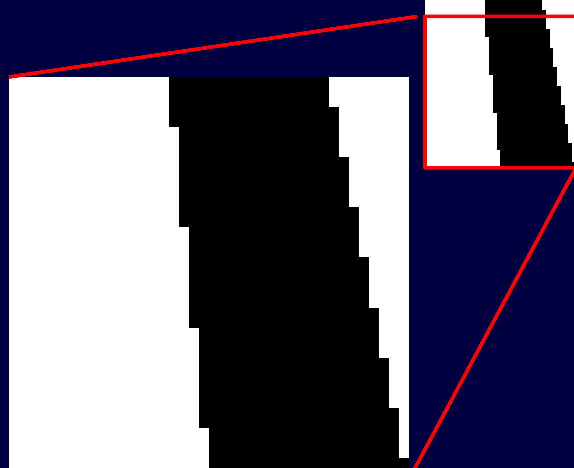
- high temporal resolution due to short intervals available
- Small amount of data due to long intervals available
- Efficient data reduction
- Conservation of statistics possible

→ Interesting Features for a Variety of Applications

Aliasing



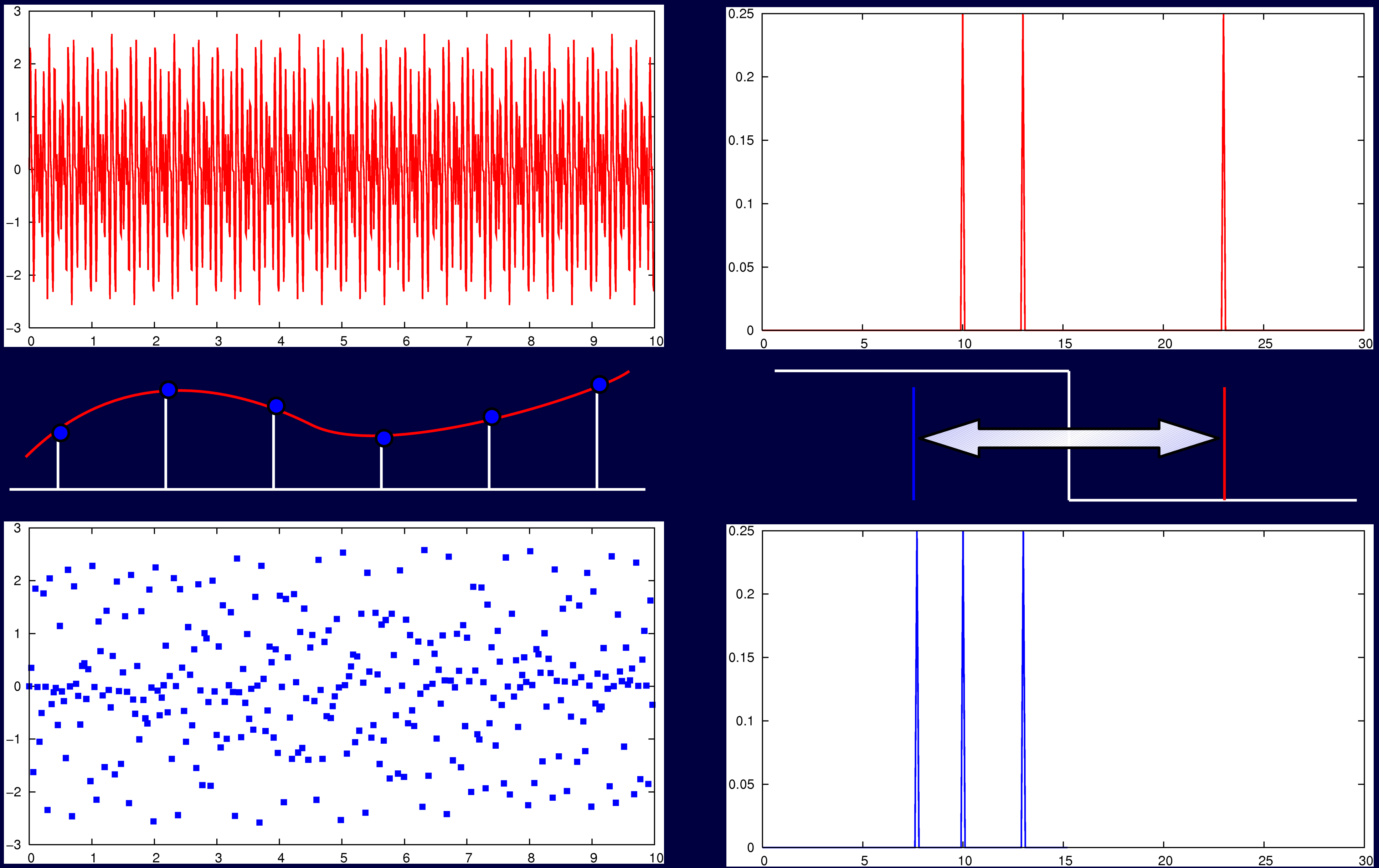




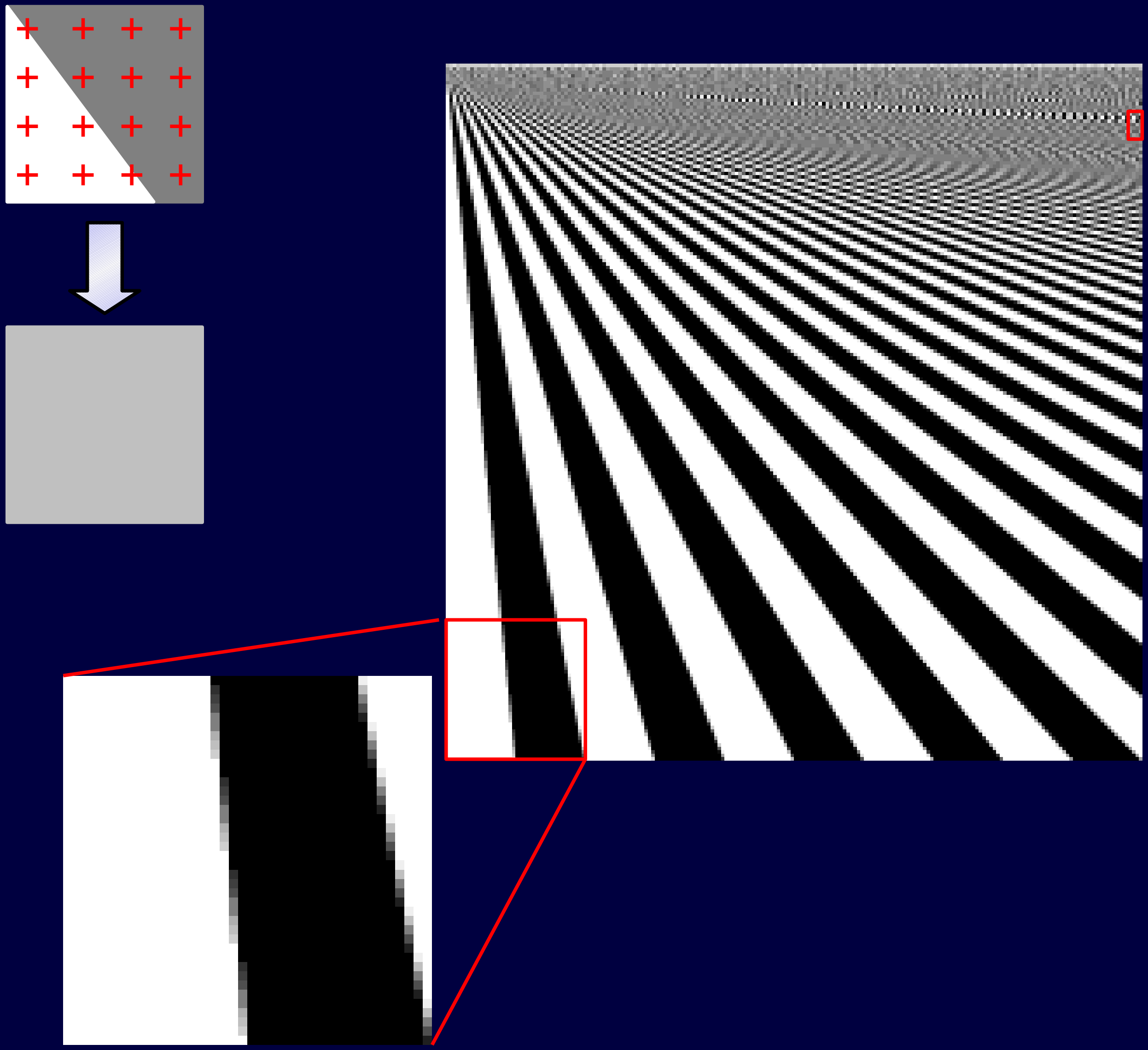
Reconstruction
of high frequencies ✗

Reconstruction
of low frequencies ✓

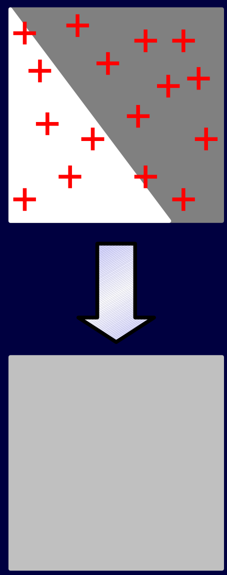
Aliasing

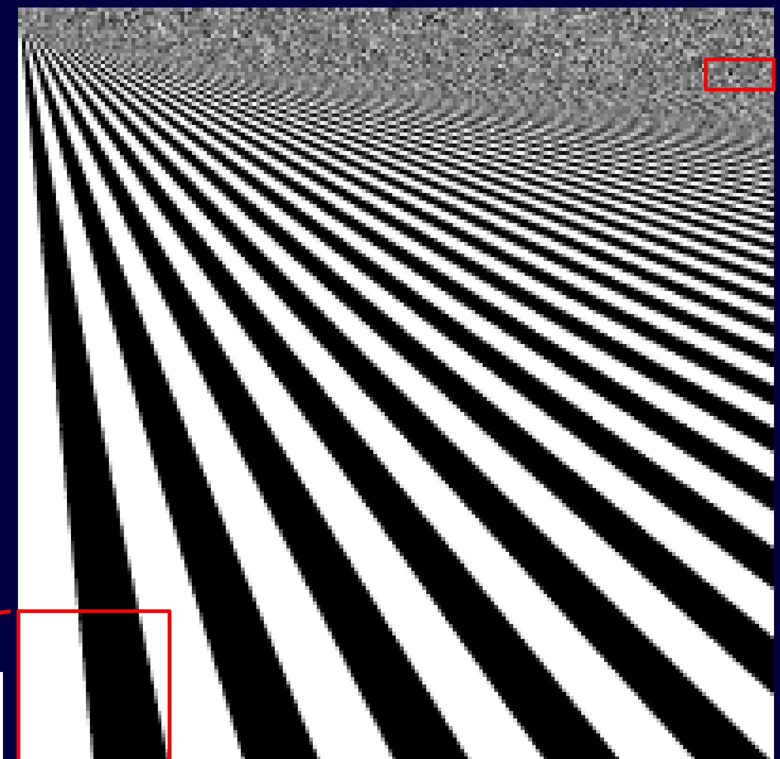


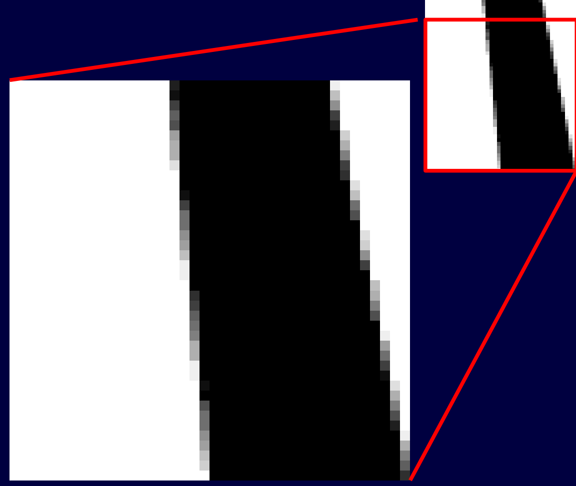
Super-Sampling



Stochastic Sampling







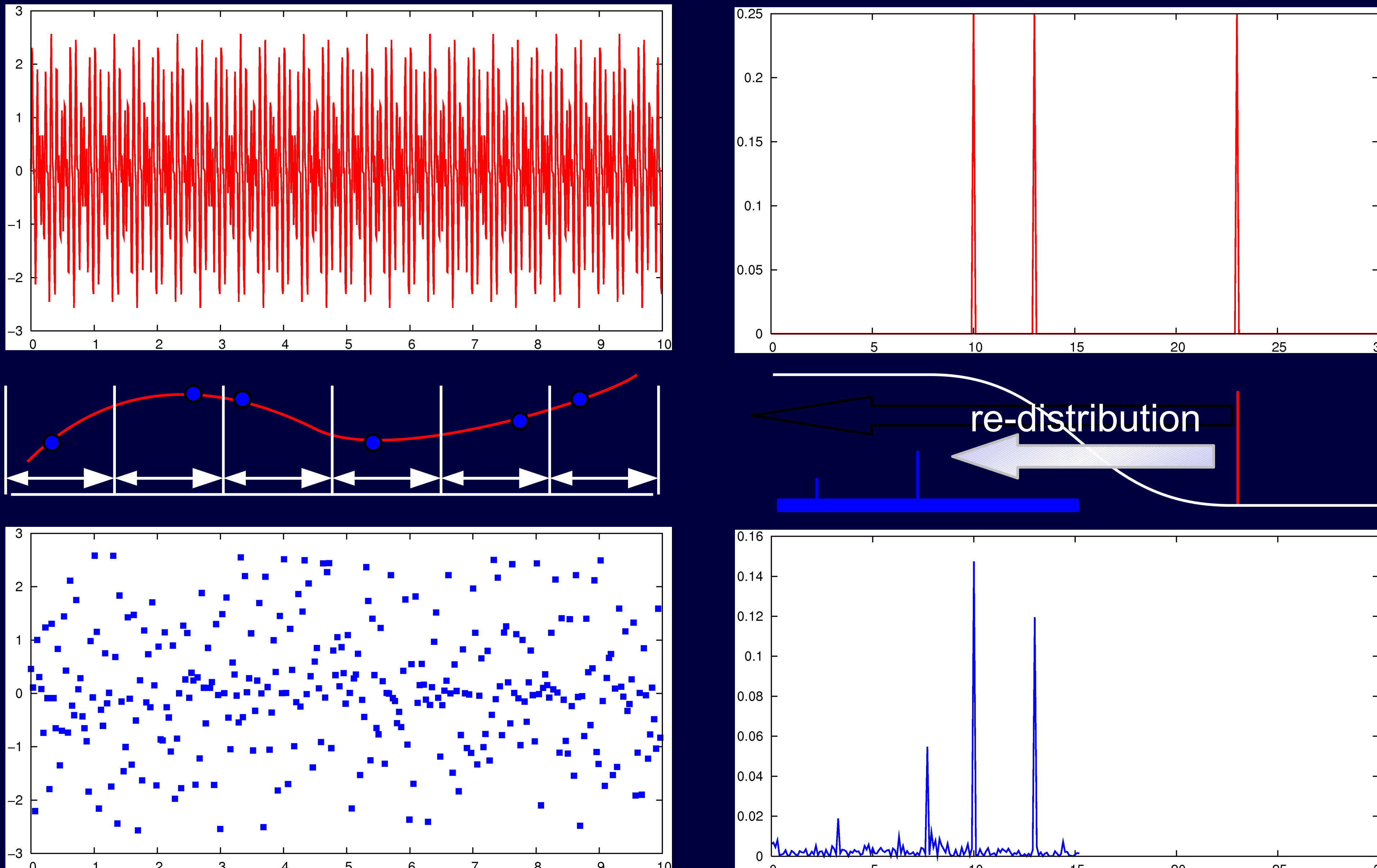
Reconstruction of high frequencies

Avoid Miss-interpretation due to aliasing

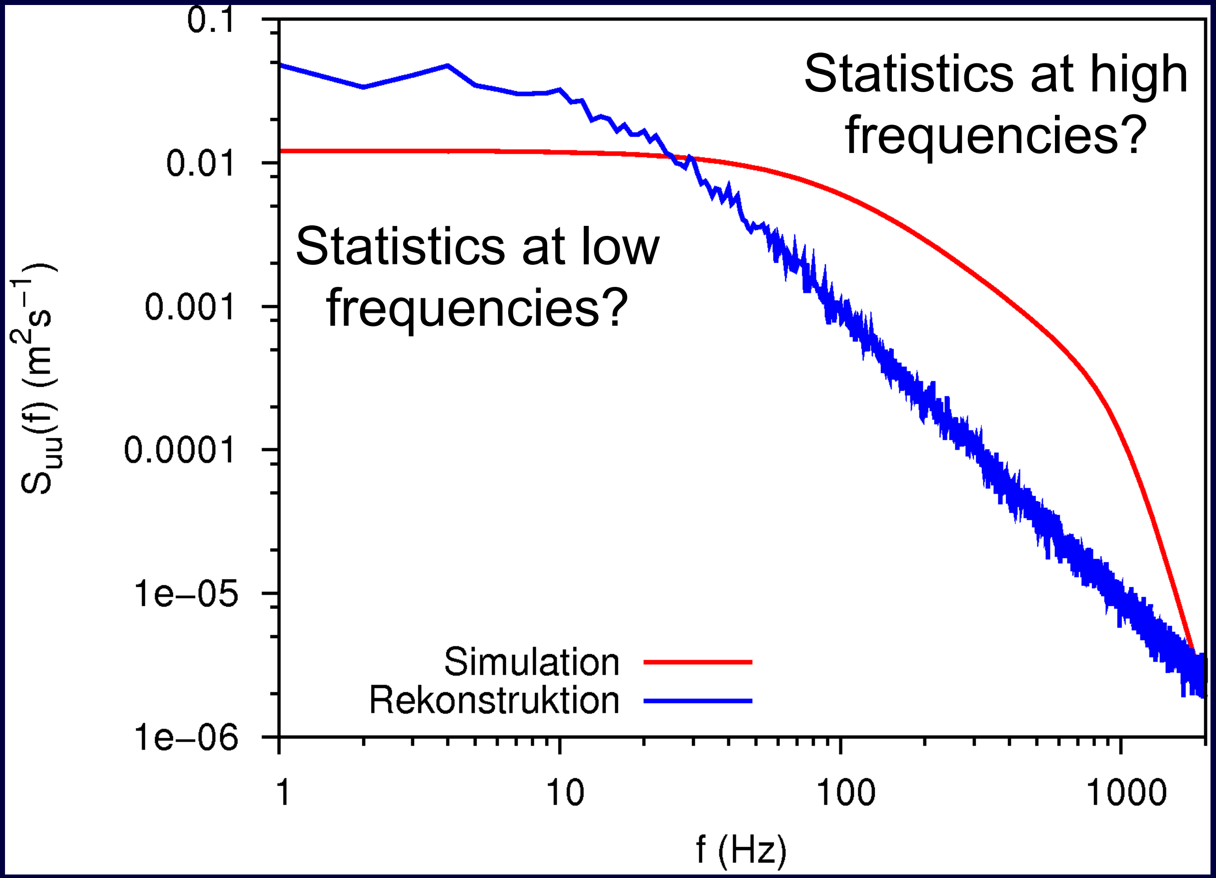
Reconstruction of low frequencies

Statistics

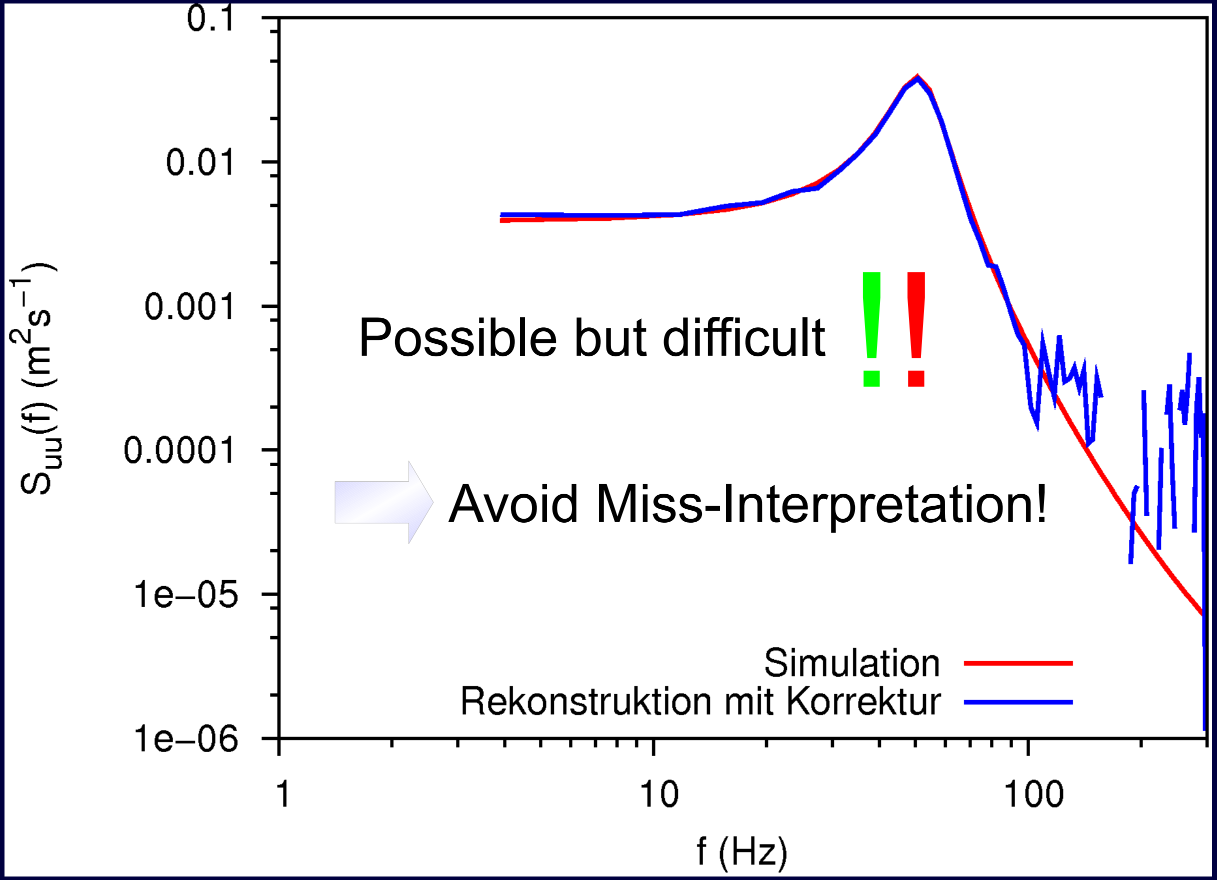
Stochastic Sampling



Stochastic Sampling

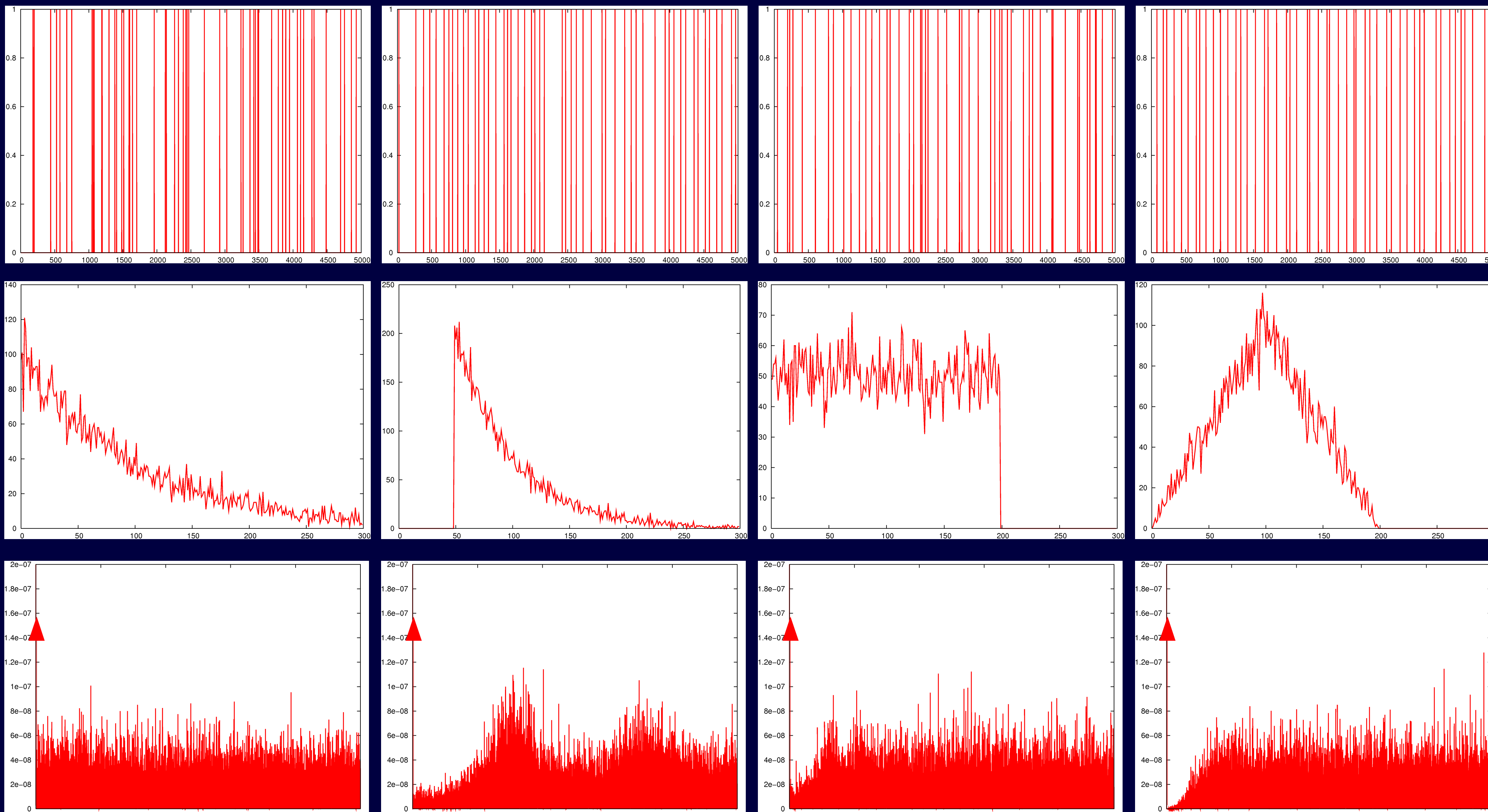


Stochastic Sampling



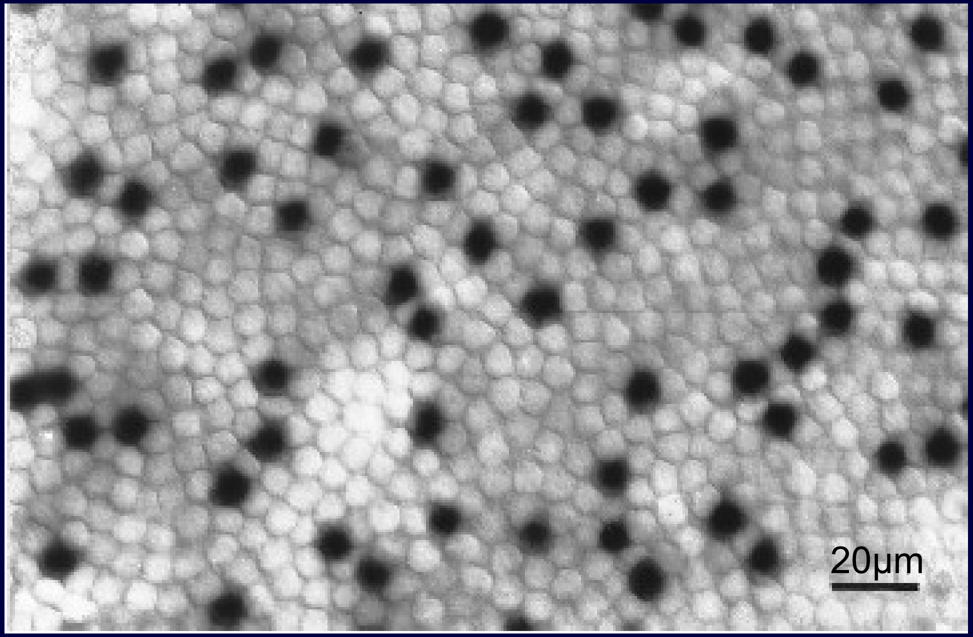
Sampling Schemes

Poisson Process | Minimum distances | Maximum Distances | Jitter



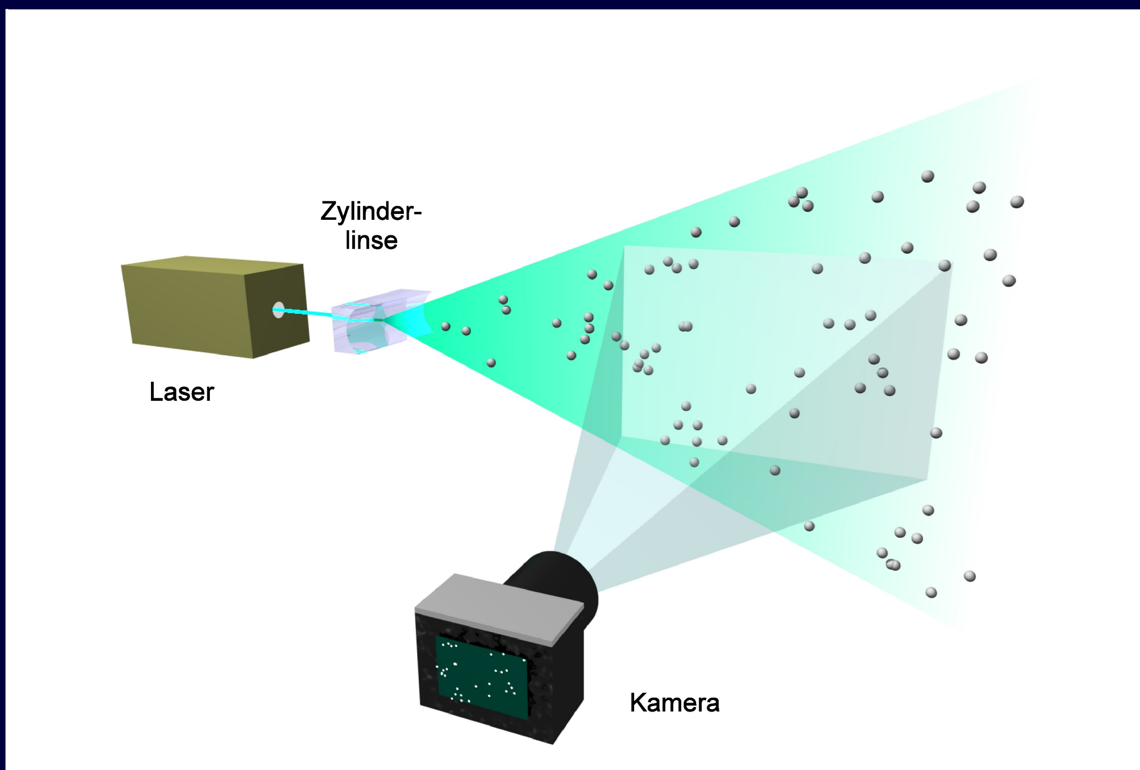
Sampling Schemes

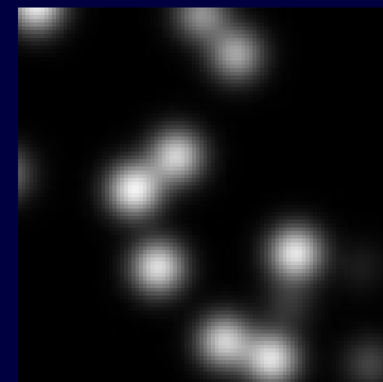
Blue cone receptors in the human retina

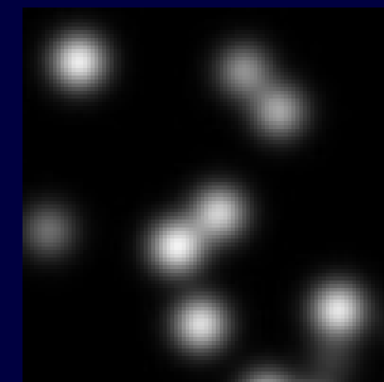


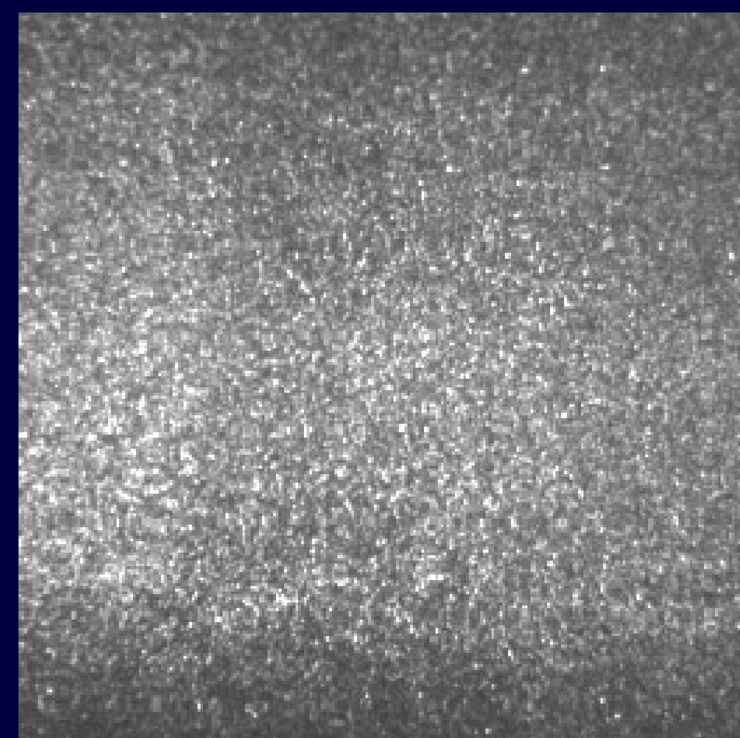
Paul R. Martin, Ulrike Grünert, Tricia L. Chan, and Keely Bumsted: Spatial order in short-wavelength-sensitive cone photoreceptors: a comparative study of the primate retina. JOSA A, Vol. 17, Issue 3, pp. 557-567

Particle Image Velocimetry









Particle Image Velocimetry